THE SCHWARZ POTENTIAL IN $\mathbb{R}^n$ AND CAUCHY'S PROBLEM FOR THE LAPLACE EQUATION

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§1. Introduction.

Let us start by recalling the concept of the Schwarz function in the complex plane. Let Γ be an analytic curve in $\mathbb{R}^2$ defined by the equation

$$\Gamma = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\} \qquad (1.1)$$

where f is a real-analytic function of x and y. We assume Γ to be free of singular points, i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not both vanish at any point of Γ. Setting $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$, where $z = x + iy$, we can rewrite (1.1) as

$$\Gamma = \{z \in \mathbb{C} : F(z,\bar{z}) = 0\}, \qquad (1.2)$$

where F is a real analytic function of z and $\bar{z}$. Then $\frac{\partial F}{\partial \bar{z}}$ does not vanish on Γ, and by the implicit function theorem we can solve (1.2) with respect to $\bar{z}$ and find a function $S(z)$ analytic in a neighborhood U of Γ such that

$$\bar{z} = S(z) \qquad (1.3)$$

holds in U. The function $S(z)$ is called the Schwarz function of the curve Γ. As an illustration of this definition we mention two simple examples: (i) $\Gamma = \mathbb{R}$, the real axis, $S(z) = z$, (ii) $\Gamma = \{z: |z| = 1\}$, the unit circle, $S(z) = \frac{1}{z}$.

For a detailed account of Schwarz functions, many interesting examples, and applications to various areas in complex function theory we refer the reader to [Da 2]. Some connections with PDE, in particular, applications to the biharmonic equation can be found in [Ga 1]. Also see [Ga 2], Ch. XVI. Applications of Schwarz functions to quadrature identities are discussed in the works of [AS], [Gu 1], [Sa 1], and [Sh 2, 4].

Let us say a word about the history of the subject, which is quite interesting. The designation "Schwarz function" is due to Philip Davis, based on its relation to the Schwarz principle of reflection. So far as we know this function does not, however, appear explicitly in the work of H.A. Schwarz. Its first appearance seems to be in a remarkable paper of G. Herglotz [Her] from 1915 which won first prize in a competition devoted to the following theme. Suppose we have a mass distribution in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and form its Newtonian (or logarithmic) potential. This function is harmonic where there is no mass, and the problem is to study its harmonic continuation into the region occupied by the mass. It is easy to see that it may indeed be harmonically continuable into a region occupied by mass, this happens e.g. if we have a uniform mass distribution in a region with an analytic boundary and this is the case studied by Herglotz, chiefly in two dimensions. (Of course, the so-continued function does not coincide with the potential in the region occupied by mass.) Herglotz introduced the "Schwarz function" of the boundary of the plane region containing the masses and observed that *the singularities of the logarithmic potential, when continued from its domain of harmonicity (i.e. the domain free of mass) into the region occupied by mass are precisely the singularities of the Schwarz function.*

The case when the Schwarz function has only polar singularities corresponds to the simplest kind of "quadrature domains" (see §4 below) or in other terms, to uniform laminae whose <u>external</u> gravitational field is identical with that produced by finitely many point masses (at the poles of the Schwarz function). Apart from the well-known case of a circular lamina, a remarkable example of this phenomenon was discovered by C. Neumann [Ne]: *if the domain bounded by an ellipse is inverted with respect to a circle concentric with the ellipse, an oval is produced which has the above property with respect to the two points into which the foci of the ellipse pass by inversion.* This example was studied by Herglotz, and related to the Schwarz function of the oval in question.

It thus appears that the subject of the Schwarz function and quadrature domains was off to a very good start in 1915. Herglotz also began the study of 3-dimensional problems, but only for bodies with an axis of symmetry. He wrote that he would return to the problem, but apparently never did so. His paper fell into oblivion and the whole subject remained dormant until it was rediscovered some 50 years later. The present work, although begun in ignorance of Herglotz' researches, may be regarded as a continuation of them.

In this paper we attempt to extend the concept of Schwarz function to $\mathbb{R}^n$. In order to find an appropriate definition let us reformulate (1.3) in terms of partial differential equations.

Since $S(z)$ is analytic near Γ, its integral is also locally analytic near Γ. Therefore, the conjugate of this integral $s(z) = U(z) + iV(z) = U(x,y) + iV(x,y)$ is locally antianalytic $(\frac{\partial}{\partial \bar{z}}(s(z)) = \overline{S(z)})$. From the Cauchy-Riemann equations

$$\frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y}, \ \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x},$$

we obtain

$$\overline{S(z)} = \frac{\partial s}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(U + iV) = \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \text{grad } U(=\nabla U).$$

Moreover, according to (1.3), on Γ

$$\nabla U = z = (x,y) \tag{1.4}$$

i.e. ∇U is equal to the identity vector field.

Since, by (1.4), $\nabla(U - \frac{1}{2}|z|^2) = 0$ on Γ, $U = \frac{1}{2}|z|^2 + \text{const}$ on Γ. Thus, summarizing, for any analytic curve Γ, there is a function U_Γ harmonic in a neighborhood of Γ and such that the equalities

$$U_\Gamma = \tfrac{1}{2}|z|^2, \qquad \nabla U_\Gamma = z$$

hold on Γ. In other words, the function U_Γ which we shall call the <u>Schwarz potential</u> of Γ, is the (unique!) solution of the following Cauchy problem posed on Γ:

$$\begin{cases} \Delta U_\Gamma = 0 \text{ near } \Gamma, \text{ where } \Delta = \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2} \text{ is the Laplace operator.} \\ U_\Gamma = \tfrac{1}{2}(x^2 + y^2) \text{ on } \Gamma, \\ \nabla U_\Gamma = (x,y) \qquad \text{on } \Gamma. \end{cases}$$

Let us introduce the following notation which we are going to use throughout the paper.

<u>Notation</u>: *For two functions* f,g *we shall write* $f \equiv g$ *on* Γ *if* $f = g$ *and* $\nabla f = \nabla g$ *on* Γ.

Making use of this notation, we can rewrite the Cauchy problem for U_Γ in the following form:

$$\begin{cases} \Delta U_\Gamma = 0 \text{ near } \Gamma \\ U_\Gamma \equiv \tfrac{1}{2}|z|^2 \text{ on } \Gamma \end{cases} \qquad (|z|^2 = x^2 + y^2,\ z = (x,y)). \tag{1.5}$$

It is clear that the problem of finding the Schwarz function $S(z)$ of Γ is equivalent to the problem of solving (1.5). This suggests the following definition.

<u>*1.1 Definition*</u>. *Let* Γ *be a regular real-analytic hypersurface in* $\mathbb{R}^n$, $n \geq 2$ *i.e.* $\Gamma = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n:$ $F(x_1, \ldots, x_n) = 0\}$, *where* F *is a real-analytic function of* $x_1, \ldots, x_n$ *and* $\nabla F \neq 0$ *on* Γ. *Then, we shall call the unique solution* U_Γ *of the following Cauchy problem posed on* Γ

$$\begin{cases} \Delta U_\Gamma = 0 \text{ "near" } \Gamma \\ U_\Gamma \equiv \frac{1}{2}|x|^2 \overset{\text{def}}{=} \frac{1}{2} \sum_{i=1}^n x_i^2 \end{cases} \tag{1.6}$$

the <u>*Schwarz*</u> <u>*potential*</u> *of the surface* Γ.

Uniqueness and existence of U_Γ for any analytic hypersurface Γ are guaranteed by the Cauchy-Kovalevskaya theorem. Thus, U_Γ is well defined.

Let us briefly discuss the contents of this paper. In §2 we find explicit formulas for the Schwarz potentials of some basic surfaces (e.g. planes, spheres, cylinders, etc.). We remark here that those computations are usually much more involved than in similar problems on the plane, where one can approach them by solving explicitly the equation (1.2) for $\bar{z}$.

In §3 we discuss the connection between Schwarz potentials and a global version of the Cauchy-Kovalevskaya theorem for the Laplace equation in the case of polynomial Cauchy data. More precisely, as has been noted by many authors, the solution of the Laplace equation $\Delta u = 0$ near an analytic curve Γ which has on Γ the Cauchy data $u \equiv P(x,y)$, where $P(x,y)$ is a polynomial, or even an entire function of x and y, can be continued analytically across Γ to any region free of singularities of the Schwarz function $S(z)$ of Γ (see [Da 2], Ch. 11, [Ga 2], Ch. XVI, [Le 1], [Le 2], [Ve] and the literature cited there). In §3 we extend this result to several particular surfaces in $\mathbb{R}^3$ and discuss the corresponding general conjecture.

In §4 we study the connection between Schwarz potentials and quadrature identities for harmonic functions. In particular, we show (Thm. 4.2, Corollary 4.3) that a domain Ω with analytic boundary Γ admits a quadrature identity if and only if the Schwarz potential of Γ extends to Ω, where modulo a term harmonic in $\bar{\Omega}$, it coincides with the potential of a distribution compactly supported in Ω. As an illustration of those results we discuss quadrature formulae for ellipsoids and cylinders in $\mathbb{R}^3$.

Finally, §5 contains additional remarks concerning the results in the paper and various open problems for future investigation.

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§2. The Schwarz potentials of some elementary surfaces.

In this section we obtain the Schwarz potentials of some elementary surfaces. First let us recall the celebrated Cauchy-Kovalevskaya theorem. We will state here a particular case of the theorem specifically related to the Laplace equation (see [Pe], Ch. 1, §2, [Ga 2], Ch. 1, §2. Ch. IV, Ch. VI, Ch. XVI, [Hö], Ch. IX).

Let $\Gamma \subset \mathbb{R}^n$, $n \geq 2$ be a non-singular analytic hypersurface. Let $f(x_1,\dots,x_n)$ be a real-analytic function in a neighborhood of Γ. Then, there exists a unique function u real-analytic in a neighborhood of Γ which solves the following initial value problem

$$\begin{cases} \Delta u = 0 & \text{near } \Gamma \\ u \equiv f & \text{on } \Gamma \end{cases}.$$

<u>Remark</u>. The usual condition that Γ be non-characteristic can be dropped here since Δ has no real characteristics. In some later discussions where Γ is taken as a <u>complex-analytic</u> variety in $\mathbb{C}^n$ we shall require the additional condition that no point of Γ be characteristic for Δ, i.e. that

$\sum_{j=1}^{n} (\frac{\partial\varphi}{\partial z_j})^2 \neq 0$ on Γ, where $\varphi = 0$ is the equation of Γ.

I. The Schwarz potential of a plane.

Let Γ be the hyperplane in $\mathbb{R}^n$

$$\Gamma : a_1x_1 + a_2x_2 + \dots + a_nx_n = c.$$

The Schwarz potential of Γ is the solution of the Cauchy problem

$$\begin{cases} U \equiv \frac{1}{2}|x|^2 \text{ on } \Gamma, \; x = (x_1,\dots,x_n) \\ \Delta U = 0 \text{ near } \Gamma. \end{cases} \tag{2.1}$$

Set $V(x_1,\dots,x_n) = U(x) - \frac{1}{2}|x|^2$. Then, $V|_\Gamma = 0$, $\nabla V|_\Gamma = 0$ and $\Delta V = -n$ near Γ. The quadratic polynomial $P(x)=(\sum_{i=1}^{n} a_ix_i - c)^2$ satisfies the same boundary conditions and $\Delta P = 2(\sum_{i=1}^{n} a_i^2)$. Therefore,

$$U(x) = \frac{1}{2}|x|^2 - \frac{n}{2|a|^2}\left(\sum_{1}^{n} a_ix_i - c\right)^2, \text{ where } a=(a_1,\dots,a_n)$$

is the unique solution of (2.1). So, the Schwarz potential of a hyperplane is a quadratic polynomial.

II. The Schwarz potential of the sphere.

Let Γ be the sphere $x_1^2 + \dots + x_n^2 = R^2$ of radius R. Let T be any rotation of $\mathbb{R}^n$ about the origin. Let U_Γ be the solution of (1.6). Then, since T commutes with the Laplacian, we observe that

$$\Delta(U_\Gamma \circ T) = \Delta U_\Gamma \circ T = 0 \text{ near } \Gamma.$$

Also, on the sphere Γ we have

$$(U_\Gamma \circ T)(x) = \tfrac{1}{2}|Tx|^2 = \tfrac{1}{2}R^2$$

$$\nabla(U_\Gamma \circ T)(x) = T^*(\nabla U(Tx)) = T^*Tx = x$$

for all $x \in \Gamma$. (Here, T^* denotes the adjoint of T). Thus, $U_\Gamma \circ T$ and U_Γ solve the same Cauchy problem on Γ. Then, from the uniqueness part of the Cauchy-Kovalevskaya theorem, we conclude that $U_\Gamma \circ T = U_\Gamma$ for all T. Hence, $U_\Gamma = U_\Gamma(|x|)$ is spherically symmetric and, therefore, has the form

$$U_\Gamma(x) = c_1 \log|x| + c_2, \; n = 2,$$

$$U_\Gamma(x) = \frac{c_1}{|x|^{n-2}} + c_2, \; n \geq 3.$$

So, from the boundary conditions (1.6) we easily find

$$U_\Gamma(x) = R^2(\log|x| + \tfrac{1}{2} - \log R), \; n = 2,$$

$$U_\Gamma(x) = -\frac{R^n}{n-2} \cdot \frac{1}{|x|^{n-2}} + \frac{n}{2(n-2)} R^2, \; n \geq 3. \qquad (2.2)$$

III. The Schwarz potential of a cylinder.

Let Γ be a cylinder $x_1^2 + x_2^2 \dots + x_{n-1}^2 = 1$. As in (I) let us represent the Schwarz potential U_Γ in the form

$$U_\Gamma(x) = \tfrac{1}{2}|x|^2 + V(x),\ V\big|_\Gamma \equiv 0,\ \Delta V = -n \text{ near } \Gamma.$$

The transformation $T: (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n + t)$, for a fixed $t \in \mathbb{R}$ maps Γ onto Γ.

We have

$$U_\Gamma(x_1,\dots,x_n + t) = \tfrac{1}{2}|x|^2 + tx_n + \frac{t^2}{2} + V(x_1,\dots,x_n + t). \quad (2.3)$$

Set $V_T = V(x_1,\dots,x_n + t)$. Since $T: \Gamma \to \Gamma$, $V_T(x_1,\dots,x_n) \equiv 0$ on Γ. Also, from (2.3) we observe that

$$\Delta V_T = -n = \Delta V \text{ near } \Gamma.$$

From the uniqueness of the solution of the Cauchy problem we conclude that $V_T = V$. Therefore, for all $t \in \mathbb{R}$

$$U_\Gamma(x_1,\dots,x_n + t) - tx_n - \frac{t^2}{2} = U_\Gamma(x_1,\dots,x_n).$$

Letting $t = -x_n$ for fixed $x_1,\dots,x_{n-1}$ we obtain that

$$U_\Gamma(x_1,\dots,x_{n-1},0) + \frac{x_n^2}{2} = U_\Gamma(x_1,\dots,x_n).$$

Since U_Γ is harmonic,

$$\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} U_\Gamma(x_1,\ldots x_{n-1},0) = -1 \text{ near } \Gamma.$$

Also, as $\nabla U\big|_\Gamma = x$ on Γ

$$\begin{cases} U(x_1,\ldots,x_{n-1},0) = \frac{1}{2}\sum_1^{n-1} x_i^2 \text{ and} \\ \frac{\partial}{\partial x_i} U(x_1,\ldots,x_{n-1},0) = x_i, \text{ on } S^{n-2}, \quad i = 1,\ldots,n-1 \end{cases}$$

where $S^{n-2} = \{(x_1,\ldots,x_{n-1}): \sum_{i=1}^{n-1} x_i^2 = 1\}$ is the $(n-2)$-dimensional sphere. Hence, $u(x_1,\ldots,x_{n-1}) = U(x_1,\ldots,x_{n-1},0)$ is the solution of the following Cauchy problem on S^{n-2}

$$\begin{cases} u \equiv \frac{1}{2}|x|_{n-1}^2 \text{ on } S^{n-2} \\ \Delta_{n-1}u = -1 \text{ near } S^{n-2}. \end{cases}$$

(Here, $\Delta_{n-1} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$, $|x|_{n-1}^2 = \sum_{i=1}^{n-1} x_i^2$). Hence, from the uniqueness of the solution of the Cauchy problem we obtain (see II)),

$$u + \frac{1}{2(n-1)} |x|_{n-1}^2 = \frac{n}{n-1} U_\gamma(x), \ x \in R^{n-1},$$

where we set $\gamma = S^{n-2}$ and U_γ denotes the Schwarz potential of γ. Then, from (2.2) we find

$$U_\Gamma(x) = \frac{n}{n-1} U_\gamma(x_1,\ldots,x_{n-1}) - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} x_i^2 + \frac{x_n^2}{2} =$$

$$= \begin{cases} \frac{3}{4}\log(x_1^2 + x_2^2) - \frac{1}{4}(x_1^2 + x_2^2) + \frac{x_3^2}{2} + \frac{3}{4}, \; n=3 \\ -\frac{n}{(n-1)(n-3)} \frac{1}{(|x|_{n-1})^{n-3}} - \frac{1}{2(n-1)}|x|_{n-1}^2 + \frac{x_n^2}{2} + \frac{n}{2(n-3)}, \; n > 3. \end{cases} \tag{2.4}$$

Let Γ be a general cylinder in $\mathbb{R}^n$, i.e. $\Gamma = \{x=(x_1,\ldots,x_n)\}$: $(x_1,\ldots,x_{n-1}) \in \gamma, x_n \in \mathbb{R}\}$, where $\gamma \subset \mathbb{R}^{n-1}$ is an analytic hypersurface. Then a similar argument yields the following.

<u>*Proposition 2.1*</u>. *The Schwarz potential* U_Γ *of the cylinder* Γ *satisfies*

$$U_\Gamma(x_1,\ldots,x_n) = \frac{n}{n-1} U_\gamma (x_1,x_2,\ldots,x_{n-1}) - \frac{1}{2(n-1)} \sum_1^{n-1} x_i^2 + \frac{x_n^2}{2}. \tag{2.5}$$

In particular, U_Γ *is analytic everywhere in* $\mathbb{R}^n$ *except possibly on the straight lines* ℓ *parallel to the* x_n *axis and passing through singularities of* U_γ, *i.e.,* $\ell = (x_1^0,\ldots,x_{n-1}^0,x_n)$, *where* $(x_1^0,\ldots,x_{n-1}^0)$ *is a singularity of* U_γ.

IV. <u>The Schwarz potential of a cone</u>.

Let $\Gamma \subset \mathbb{R}^n$, $n \geq 3$ be a cone $x_1^2 + \ldots + x_{n-1}^2 = x_n^2$. Let

$\tau: \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\tau(x)=tx$, where $t \in \mathbb{R}$ is fixed. Then $\tau: \Gamma \to \Gamma$. Again, we decompose the Schwarz potential U of Γ as

$$U_\Gamma(x) = \tfrac{1}{2}|x|^2 + V(x),$$

where

$$V \equiv 0 \text{ on } \Gamma$$

$$\Delta V = -n \text{ near } \Gamma.$$

Then,

$$U_\Gamma \circ \tau(x) = \frac{t^2}{2}|x|^2 + V(tx)$$

or

$$\frac{1}{t^2} U_\Gamma \circ \tau(x) = \tfrac{1}{2}|x|^2 + \frac{1}{t^2} V(tx).$$

Since

$$\frac{1}{t^2} V(tx) \equiv 0 \text{ on } \Gamma \quad \text{and}$$

$$\Delta\left(\frac{1}{t^2} V(tx)\right) = (\Delta V)(tx) = -n \text{ near } \Gamma,$$

from the uniqueness part of the Cauchy-Kovalevskaya theorem we conclude that

$$\frac{1}{t^2} V(tx) = V(x)$$

and therefore,

$$\frac{1}{t^2} U_\Gamma \circ \tau(x) = U_\Gamma(x). \tag{2.6}$$

Let $T: S^{n-2} \to S^{n-2}$ be a rotation of the unit sphere in $\mathbb{R}^{n-1}$. Then, $T = T \times \mathrm{Id}: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (T(x_1,\ldots,x_{n-1}),x_n)$ leaves the cone Γ invariant. The same calculation as in (II) shows that $U \circ T$ solves the same Cauchy problem on Γ and, therefore,

$$U_\Gamma \circ T = U_\Gamma .$$

So, we can regard U_Γ as a function of two variables $r^2 = \sum_{i=1}^{n-1} x_i^2$ and x_n. Taking $t = \frac{1}{x_n}$ in (2.6), we obtain

$$U_\Gamma = U_\Gamma(r^2,x_n) = x_n^2\, U_\Gamma(\frac{r^2}{x_n^2},1).$$

Introducing new variables $s = x_n^2$, $t = \frac{r^2}{x_n^2}$, we have

$$U_\Gamma(x_1,\ldots,x_n) = s\,\phi(t), \tag{2.7}$$

where $\phi(t) = U_\Gamma(\frac{r^2}{x_n^2},1) = \frac{1}{x_n^2}\, U_\Gamma(x_1,\ldots,x_n)$.

Applying the chain rule we compute

$$\frac{\partial U_\Gamma}{\partial x_i} = \frac{\partial U_\Gamma}{\partial s}\frac{\partial s}{\partial x_i} + \frac{\partial U_\Gamma}{\partial t}\frac{\partial t}{\partial x_i} = s\phi'(t)\frac{2x_i}{x_n^2} = 2x_i\phi', \quad i=1,\ldots,n-1.$$

$$\frac{\partial U_\Gamma}{\partial x_n} = \frac{\partial U_\Gamma}{\partial s}\frac{\partial s}{\partial x_n} + \frac{\partial U_\Gamma}{\partial t}\frac{\partial t}{\partial x_n} = 2x_n\phi(t) - \phi'(t)\frac{2r^2}{x_n}. \tag{2.8}$$

Differentiating (2.8) one more time we obtain

$$\frac{\partial^2 U_\Gamma}{\partial x_i^2} = 2\phi'(t) + \frac{4x_i^2}{x_n^2}\phi''(t), \quad i = 1,\ldots,n-1$$

$$\frac{\partial^2 U_\Gamma}{\partial x_n^2} = 2\phi(t) + 2x_n\phi'(t)(-\frac{2r^2}{x_n^3}) + \frac{2r^2\phi'}{x_n^2} -$$

$$-\frac{2r^2}{x_n}\phi''(t)(-\frac{2r^2}{x_n^3}) = 2\phi - 2t\phi' + 4t^2\phi''.$$

Thus, $\Delta U_\Gamma = 0$ transforms into the following linear homogeneous equation of the second order

$$(4t^2 + 4t)\phi'' + [2(n-1) - 2t]\phi' + 2\phi = 0,$$

or (2.9)

$$(t^2 + t)\phi'' + [\frac{(n-1)}{2} - \tfrac{1}{2}t]\phi' + \tfrac{1}{2}\phi = 0.$$

The boundary conditions on Γ yield, in view of (2.8),

$$U_\Gamma\big|_\Gamma = s\phi(t)\big|_{t=1} = x_n^2 = s,$$

$$\nabla U_\Gamma\big|_\Gamma = (x_1,\ldots,x_n) = (2x_1\phi'(1),\ldots,2x_n\phi(1) - 2x_n\phi'(1)).$$

Hence,

$$\phi(1) = 1 \text{ and } \phi'(1) = \tfrac{1}{2}. \qquad (2.10)$$

Observe that $\phi_1(t) = \frac{n-1}{2} - \frac{1}{2}t$ is a solution of (2.9). Therefore, (see [In], Ch. XII) the second solution of (2.9) has the form

$$\phi_2(t) = \phi_1\left\{ \int \phi_1^{-2} \exp\left\{-\tfrac{1}{2}\int (n-1-t)(t^2+t)^{-1}dt\right\}dt\right\}. \qquad (2.11)$$

So, for $0 < t \leq 1$, $\phi(t) = C_1\phi_1(t) + C_2\phi_2(t)$, where the constants C_1, C_2 can be determined from (2.10). In particular, for $n = 3$ we obtain

$$\exp\{-\int (1-t/2)[t(t+1)]^{-1}dt\} = t^{-1}(t+1)^{3/2}$$

and

$$\int (t+1)^{3/2}t^{-1}(1-t/2)^{-2}dt = \ell n\frac{u-1}{u+1} - \frac{6u}{u^2-3}, \text{ where } t = u^2 - 1.$$

Therefore, from (2.11) we compute

$$\phi_2 = \left(1-\frac{u^2-1}{2}\right)\left(\ell n\frac{u-1}{u+1} - \frac{6u}{u^2-3}\right) = \tfrac{1}{2}(3-u^2)\ell n\frac{u-1}{u+1} + 3u.$$

So,

$$\phi = C_1(3-u^2) + C_2[\tfrac{1}{2}(3-u^2)\ell n\frac{u-1}{u+1} + 3u]$$

and

$$U_\Gamma(x) = C_1 x_3^{\,2}(3-(|x|/x_3)^2) + C_2 x_3^2\left\{\frac{1}{2}(3-(|x|/x_3)^2)\ln\left(\frac{|x|-x_3}{|x|+x_3}\right) + \right.$$

$$\left. + 3(|x|/x_3)\right\};$$

$$x=(x_1,x_2,x_3),\ |x|=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}},\ x_3 > 0.$$

Since $\phi'_u = \phi'_t \cdot 2u$ and $t=1$ as $u = \sqrt{2}$ we determine the constants C_1, C_2, from the boundary conditions (2.10),

$$C_1 = \frac{8 - 3\sqrt{2} - (4\sqrt{2} + 1)\ \ln(\sqrt{2} + 1)}{32},\quad C_2 = \frac{4\sqrt{2} + 1}{16}.$$

Remark. (2.12) is the Schwarz potential of the upper nappe of the cone Γ. It is analytic everywhere in $\mathbb{R}^3$ except for the x_3-axis. The Schwarz potential of the lower nappe can be obtained if one replaces x_3 in (2.12) by $|x_3| = -x_3$. So it is also analytic everywhere in $\mathbb{R}^3$ except x_3-axis. However, analytic continuition of (2.12) into the lower half-space $\{x_3 < 0\}$ is not equal to the Schwarz potential of the lower nappe. Therefore $\{x_3 = 0\}$ is a convenient place to put a "branch cut" in order to avoid multivaluedness of the Schwarz potential of the whole cone. So, we can say that U_Γ is regular everywhere in $\mathbb{R}^3$ except for the x_3-axis and the plane $\{x_3 = 0\}$.

The same holds for $n > 3$ (cf. (2.7), (2.11), (2.12)) and, in fact, as we shall see in §3, solutions of all Cauchy's problems for the Laplace equation with polynomial Cauchy data on the cone Γ also extend analytically to the same domain.

§3. Schwarz potentials and Cauchy's problem for the Laplace equation.

In spite of the generality of the Cauchy-Kovalevskaya theorem, what makes this theorem difficult to work with in certain concrete situations, is the fact that it does not precisely specify the region of analyticity of the solution u in terms of geometric properties of Γ and a region of analyticity of f.

The following cute result filling this gap for $\mathbb{R}^2$ has been noted in various forms by different authors, e.g. see [Da 2], Ch. 11, [Ga 2], Ch. XVI, [Le 1], [Le 2], [Ve, Ch. I]. Also, one can refer to an excellent survey by P. Henrici [Hen].

Theorem 3.1. Let Γ be an analytic curve in $\mathbb{C}$ whose Schwarz function $\bar{z} = S(z)$ is analytic in the domain Ω. Let $f = f(x,y)$ be a real-entire function of x and y, i.e. the restriction to $\mathbb{R}^2$ of an entire function on $\mathbb{C}^2$. Then, the solution of the initial value problem

$$\begin{cases} \Delta u = 0 \text{ near } \Gamma \\ u \equiv f \text{ on } \Gamma \end{cases} \tag{3.1}$$

can be analytically continued throughout Ω.

For the reader's convenience we outline a simple proof of the theorem.

Proof. Note that the uniqueness and existence of the solution u of (3.1) follow directly from the Cauchy-Kovalevskaya theorem. So, let u be a local solution of (3.1) near Γ. Since, $\Delta u = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} = 0$, the function $\frac{\partial u}{\partial \bar{z}}$ is antianalytic near Γ. Also, according to (3.1), we have on Γ

$$u_{\bar{z}}\Big|_{\Gamma} = \frac{\partial u}{\partial \bar{z}}(z,\bar{z})\Big| = f_{\bar{z}}(z,\bar{z})\Big|_{\Gamma} = f_{\bar{z}}(\overline{S(z)},\bar{z}). \qquad (3.2)$$

Since $f(z,w)$ is analytic with respect to (z,w) in $\mathbb{C}^2$, $f(\overline{S(z)},\bar{z})$ and therefore $f_{\bar{z}}(\overline{S(z)}\bar{z})$ are antianalytic in Ω. Then, from (3.2) we derive that $u_{\bar{z}}$ can be continued as an antianalytic function throughout Ω, with (3.2) holding everywhere in Ω. Similarly, for the analytic function u_z we have on Γ

$$u_z\Big|_{\Gamma} = f_z(z,\bar{z})|_{\Gamma} = f_z(z,S(z)).$$

It is obvious that the function $f(z,S(z))$ is analytic in Ω. Therefore, $f_z(z,S(z))$ is also analytic throughout Ω. So u_z can be continued into Ω as analytic function $f_z(z,S(z))$. Thus, both derivatives $\frac{\partial u}{\partial x} = (\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}})$, $\frac{\partial u}{\partial y} = \frac{1}{i}(\frac{\partial u}{\partial \bar{z}} - \frac{\partial u}{\partial z})$ can be continued as real analytic functions into Ω. This yields that u can also be continued as a real analytic function throughout Ω.

To illustrate this result let us mention here a few examples where this global Cauchy-Kovalevskaya theorem has a particularly explicit form.

As above, let f be an entire real analytic function of x

and y in $\mathbb{R}^2$.

Corollary 3.2. Let Γ *be a simple closed analytic curve whose Schwarz function* $S(z)$ *is meromorphic inside* Γ. *Then, the solution of (3,1) can be continued as a real analytic function into the whole interior of* Γ *excluding the poles of* $S(z)$. *In particular, (i) if* Γ *is a straight line, then the solution of (3.1) is an entire real analytic function on the whole plane (cf. [Hö, Ch. IX, Lemma 9.1.4]); (ii) if* Γ *is a circle, the solution of (3.1) can be analytically continued to the whole plane excluding the center of the circle.*

Remark. We recall, that the condition of $S(z)$ being meromorphic in the interior G of Γ is equivalent to G being a so called "quadrature domain", i.e.

$$\iint_G u \, dxdy = \sum_{i=1}^{m} \sum_{j=0}^{n_i} c_{ij} \, u^{(j)}(z_i)$$

for all functions u analytic in G. Here, $\{z_i\}_1^n$ are the poles of $S(z)$ (see [Da 2, Ch. 14], [AS], [Sa 1], [Sh 4]).

Corollary 3.3. Let Γ *be a conic section (i.e. ellipse, hyperbola or parabola), then solutions of (3.1) can be analytically continued to the entire plane excluding the foci of* Γ.

This corollary follows from Theorem 3.1, since one can easily find an explicit expression for $S(z)$ which yields that

$S(z)$ only has algebraic singularities at the foci of Γ.

One of the major objectives of this research was an attempt to establish at least a partial generalization of Theorem 3.1 to $\mathbb{R}^n$, by proving the following conjecture.

<u>Conjecture 3.4</u>. *Let* Γ *be an analytic hypersurface in* $\mathbb{R}^n$ *whose Schwarz potential* $U_\Gamma(x)$ *is real analytic in the domain* Ω. *Then, for any polynomial* $P(x) = P(x_1,\ldots,x_n)$ *the solution of the initial value problem (3.1) with* $f = P$ *can be continued analytically into* Ω.

If this conjecture does hold, it would be a strong indication that the notion of the Schwarz potential in $\mathbb{R}^n$, $n \geq 3$, is legitimate and the definition we have given in §1 is a natural one.

Let us try to give a heuristic explanation as to why we think Conjecture 3.4 (perhaps with some restrictions) seems plausible. Here, it is more convenient to think of Γ in $\mathbb{C}^n$ rather than $\mathbb{R}^n$. So, let Γ be a nonsingular analytic hypersurface in $\mathbb{C}^n$, i.e., for each $z^0 \in \Gamma$ there is a neighborhood $U(z^0)$ of z^0 in $\mathbb{C}^n$ and a function φ holomorphic in $U(z^0)$ such that $(\frac{\partial\varphi}{\partial z_j})(z^0)$ are not all 0 and

$$\Gamma \cap U(z^0) = \{z \in U(z^0) : \varphi(z) = 0\}.$$

By the Cauchy-Kovalevskaya Theorem, the Cauchy problem

$$\begin{cases} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j^2} = f \\ u \equiv 0 \text{ on } \Gamma \end{cases} \tag{3.1*}$$

has, for each f holomorphic on a neighborhood of z^0, a solution u holomorphic on some (perhaps smaller) neighborhood of z^0, provided Γ is not characteristic at z^0 for the operator $L = \sum_{j=1}^{n} \left(\frac{\partial}{\partial z_j}\right)^2$, i.e., provided $\sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial z_j}\right)^2$ does not vanish at z^0. Our heuristic argument rests on two considerations.

(a) If $f(z^0) \neq 0$ and Γ is characteristic at z^0, then the Cauchy problem (3.1*) has no solution holomorphic on a neighborhood of z^0 (in particular, this is so for $f \equiv 1$, so <u>the Schwarz potential</u> $U_\Gamma = \frac{1}{2}\left(\sum_{j=1}^{n} z_j^2\right) - nu$ <u>is singular on the whole set Char (Γ) of points where</u> Γ <u>is characteristic</u>.)

To prove this, suppose u is a solution of (3.1*), holomorphic on a neighborhood of z^0. By the Weierstrass preparation theorem φ can be factored into pseudoprimes near z^0, and there is no loss of generality to assume φ itself is a pseudoprime. Since u vanishes where φ does, we have then

$$u = \varphi v$$

for some v holomorphic in a neighborhood U of z^o. Hence, in U

$$\frac{\partial u}{\partial z_j} = \varphi \frac{\partial v}{\partial z_j} + \left(\frac{\partial \varphi}{\partial z_j}\right) v \quad (j=1,\ldots,n)$$

and since $\frac{\partial u}{\partial z_j}$ vanishes in $U \cap \Gamma$, so does $\left(\frac{\partial \varphi}{\partial z_j}\right) v$ for each $j=1,\ldots,n$ and hence (taking U small enough, and using the non-singularity of Γ), $v = \varphi\omega$ for some ω holomorphic near z^o. Hence $u = \varphi^2\omega$, and applying the operator L to both sides gives, observing (3.1*)

$$f = (L\,\varphi^2)\omega + 2 \sum_{j=1}^{n} \left(\frac{\partial \varphi^2}{\partial z_j}\right)\left(\frac{\partial \omega}{\partial z_j}\right) + \varphi^2\, L\omega .$$

Now,

$$L\,\varphi^2 = 2\varphi(L\varphi) + 2 \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial z_j}\right)^2 ,$$

so we get

$$f = 2 \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial z_j}\right)^2 \omega + \varphi\, g$$

where g is holomorphic near z^o. This shows that if $f(z^o) \neq 0$, Γ is not characteristic at z^o.

(b) The second consideration is the feeling that all singularities that arise by continuing analytically the solution u of (3.1*) from Γ into $\mathbb{C}^n$ must somehow "come from" singularities on Γ, more precisely from points of Char (Γ), and that the manner of propagation of singularities through $\mathbb{C}^n$ depends only on the operator L and the manifold Γ, not the particular data f.

Thus, the argument goes, the Schwarz potential must encounter <u>all</u> singularities in $\mathbb{C}^n$ that any solution u of (3.1*) does, coming from data f that is holomorphic on a neighborhood of Γ (assuming f is a <u>polynomial</u> is just a very convenient way to assure that it is holomorphic on a neighborhood in $\mathbb{C}^n$ of Γ. Observe that even the unit sphere of $\mathbb{R}^n$, when "complexified" to the subset $\{\sum z_j^2 = 1\}$ of $\mathbb{C}^n$ is unbounded).

This argument can be made perfectly rigorous in $\mathbb{C}^2$, since then the propagation of singularities is very transparent, they move along the complex bicharacteristic lines $z_1 + iz_2 = \text{const.}$, $z_i - iz_2 = \text{const.}$, and indeed that is exactly what the proof of Theorem 3.1 amounted to (although couched in slightly different language).

However, in $\mathbb{C}^n$ for $n \geq 3$, such propagation of singularities is geometrically much more subtle and the picture is much less clear than in $\mathbb{C}^2$. What is known, based on work of Zerner [Ze] and Bony-Schapira [BS] is that if Lu is holomorphic in a domain D of $\mathbb{C}^n$, $z^0 \in \partial D$ and a portion of the boundary of D near z^0 is given by an equation

$$\psi(x_1,y_1,\ldots,x_n,y_n) = 0$$

where ψ is a real function of class C^1 on a neighborhood of $z^o = (x_1^o + iy_1^o,\ldots,x_n^o + iy_n^o)$ and

$$\sum_{j=1}^{n} \left(\frac{\partial\psi}{\partial x_j} - i\,\frac{\partial\psi}{\partial y_j}\right)^2 = 4 \sum_{j=1}^{n} \left(\frac{\partial\psi}{\partial z_j}\right)^2 , \quad z_j=x_j+iy_j$$

does not vanish at z^o, then u is holomorphically extendible across z^o. (This statement is a specialization to L of a theorem valid for general linear partial differential operators with analytic coefficients.) Thus, the known general criterion for continuability of solutions of $L u = f$ in $\mathbb{C}^n$ is a local one, in terms of a differential-geometric condition at the boundary of an already known domain of regularity, rather than in terms of propagation along some class of paths that can be specified <u>a priori</u>. Using the above criterion (or alternatively, by another method based on Fourier analysis as done by Kiselman [Ki]) one can give a reasonably explicit necessary and sufficient condition for continuation of solutions of $Lu = f$ from a <u>convex</u> set in $\mathbb{C}^n$ to a larger <u>convex</u> set. (Gunnar Johnsson [Jo 1] has, by further elaborating the continuity method used by Hörmander [Hö, §9.4] (based upon the Zerner-Bony-Schapira ideas) obtained a few <u>special</u> results concerning nonconvex sets. For example he has shown [Jo 2] that a solution of $Lu = 0$ holomorphic on a neighborhood of $\{z: \sum z_j^2 = 1\}$ extends holomorphically to the complement in $\mathbb{C}^n$

of the variety $\{z: \sum z_j^2 = 0\}$ which is everywhere characteristic for L).

The interest of Conjecture 3.4 (which has an obvious generalization to linear PDE with holomorphic coefficients) is that it involves a fundamental issue of how singularities propagate in $\mathbb{C}^n$. To our knowledge the concept to use a "test solution" (like the Schwarz potential) to "sniff out" all the possible singularities for Cauchy's problem has not been investigated before.

Unfortunately, we have been unable to prove Conjecture 3.4 in general. In what follows we verify this conjecture for the special cases of surfaces considered in §2. This involves explicitly solving certain Cauchy problems for the Laplace equation. These results appear to be new, and perhaps have an independent interest.

For the sake of clarity and in order to simplify the notation, we shall conduct the arguments for $n = 3$.

Proposition 3.5 *Let* Γ *be the plane whose equation is* $a_1x_1 + a_2x_2 + a_3x_3 - c = 0$, *and let* $P = P(x_1,x_2,x_3)$ *be a polynomial of degree* n. *Then, the solution of the initial value problem*

$$\begin{cases} u \equiv P \text{ on } \Gamma \\ \Delta u = 0 \text{ near } \Gamma \end{cases} \tag{3.3}$$

is a harmonic polynomial of a degree less or equal to n. *(cf. to (I) in §2).*

<u>Remark</u>. Proposition 3.5 is well-known, even in the more general case of entire Cauchy data, and can be obtained if one imitates the proof of the general Cauchy-Kovalevskaya theorem and observes that the power series determining the solution near Γ, in fact, converges in the entire space (see [Hö, Ch. IX, §1], [Ga 2, Ch. 16, §1]). However, here we present a different proof specifically for polynomial data.

Proof. Let $v = u - P$. Then, $v|_{\Gamma} \equiv 0$, $\Delta v = -\Delta P$ near Γ and conversely, if v satisfies this, $u = v + P$ solves the original problem. Thus, it is enough to show the latter Cauchy problem is solvable for a polynomial v of degree at most n. For any polynomial Q, the polynomial

$$v: = \left(\sum_{i=1}^{3} a_i x_i - c\right)^2 Q \equiv 0 \text{ on } \Gamma.$$

Hence, our proposition follows immediately from the following assertion.

Lemma 3.6. Let P_n = {polynomials in x_1, x_2, x_3 of degree $\leq n$}. *Then, the linear mapping* $T: P_n \to P_n$, $P_n \ni p \to \Delta[(\sum_{i=1}^{3} a_i x_i - c)^2 p]$ *is surjective.*

To prove the lemma we observe that since $\dim P_n < +\infty$ it suffices to show that the mapping T is injective. But if

$p \in \ker T$, then the polynomial $q(x) = (\sum_{i=1}^{3} a_i x_i - c)^2 p(x)$ solves the initial value problem (3.3) for $P = 0$. Hence, according to the uniqueness part of the Cauchy-Kovalevskaya theorem $q(x) = 0$, and, therefore, $\ker T = \{0\}$.

<u>*Proposition 3.7*</u>. *Let* Γ *be a sphere.* $\Gamma: \sum_1^3 x_i^2 = r^2$ *and let* $P = P(x_1,x_2,x_3)$ *be a polynomial. Then, the solution* u *of the initial value problem (3.3) can be analytically continued to* $\mathbb{R}^3\setminus\{0\}$. *Moreover,* u *can be expressed as a finite linear combination of functions of the form* H_n *and* $\frac{H_n}{|x|^{2n+1}}$, $n = 0,1,\dots,$ *where* $H_n \in P_n$ *denote homogeneous harmonic polynomials of degree* n.

<u>*Proof*</u>. Without loss of generality we can assume that $r = 1$. We shall need the following lemma which is known ([SW, Ch. II]).

<u>*Lemma 3.8*</u>. *Let* $Q \in P_n$. *Then, there exists a harmonic polynomial* $H_n \in P_n$ *such that* $H_n|_\Gamma = Q|_\Gamma$. *Moreover,*

$$Q = H_n + (\sum_1^3 x_i^2 - 1)\ R,\ R \in P_{n-2}.$$

<u>*Proof*</u>. It suffices to show that there exists $R \in P_{n-2}$: $\Delta[(x_1^2+x_2^2+x_3^2-1)R] = \Delta Q$, because then we can define

$H_n = Q - (x_1^2+x_2^2+x_3^2-1)R$. As in the proof of Lemma 3.6, it is enough to show that the mapping $T : \mathcal{P}_n \to \mathcal{P}_n$, $T(P) = \Delta[(x_1^2+x_2^2+x_3^2-1)P]$ is surjective, and since $\dim\mathcal{P}_n < +\infty$ this would follow if we could show that T is injective. Let $P \in \ker T$. Then, $(x_1^2+x_2^2+x_3^2-1)P$ is harmonic in the interior of Γ and vanishes on Γ. So, by the maximum principle it is identically zero, and therefore $\ker T = \{0\}$.

The following lemma is also known (see [Ke, p. 233]) and can be verified by straightforward calculations. We shall omit the proof.

Lemma 3.9. Let H_n *be a homogeneous harmonic polynomial of degree* n. *Then,* $\dfrac{H_n(x)}{|x|^{2n+k-2}}$ *is harmonic in* $\mathbb{R}^k\setminus\{0\}$.

Let us proceed with the proof of Proposition 3.7. Iterating Lemma 3.8, we obtain that for any polynomial $P \in \mathcal{P}_n$ there exist harmonic polynomials $H_n \in \mathcal{P}_n$, $H_{n-2} \in \mathcal{P}_{n-2}$ and a polynomial $R \in \mathcal{P}_{n-4}$ such that

$$P = H_n + (x_1^2+x_2^2+x_3^2-1)H_{n-2} + (x_1^2+x_2^2+x_3^2-1)^2R \tag{3.4}$$

Hence, in view of (3.4), the Cauchy data induced by P on Γ has the form (using an obvious notation for Cauchy data)

$$\begin{pmatrix} P \\ \nabla P \end{pmatrix} = \begin{pmatrix} H_n \\ \nabla H_n \end{pmatrix} - 2\begin{pmatrix} 0 \\ x_1H_{n-2}, x_2H_{n-2}, x_3H_{n-2} \end{pmatrix}, \tag{3.5}$$

Since solutions of (3.3) depend linearly on the initial data and H_n is the obvious unique solution of $\Delta u = 0$ with the Cauchy data $\begin{pmatrix} H_n \\ \nabla H_n \end{pmatrix}$, it remains for all $n \geq 1$ to solve the problem $\Delta u = 0$ with the initial data $\begin{pmatrix} 0 \\ x_1 H_n, x_2 H_n, x_3 H_n \end{pmatrix}$ on Γ, where H_n is a homogeneous harmonic polynomial of degree n (since every harmonic polynomial can be represented as a sum of homogenous harmonic polynomials). Observe that

$$H_n \Big|_\Gamma = \frac{H_n}{|x|^{2n+1}}\Big|_\Gamma \text{ and}$$

$$\nabla\left(\frac{H_n}{|x|^{2n+1}}\right)\Big|_\Gamma = -(2n+1)\, H_n x + \nabla H_n\Big|_\Gamma, \quad x = (x_1, x_2, x_3).$$

Therefore, according to Lemma 3.9 for k=3 the solution of the Laplace equation with the Cauchy data $\begin{pmatrix} 0 \\ x_1 H_n, x_2 H_n, x_3 H_n \end{pmatrix}$ on Γ is precisely

$$-\frac{1}{2n+1}\left(\frac{H_n}{|x|^{2n+1}} - H_n\right)$$

and the Proposition is proved.

To discuss a corresponding version of Proposition 3.7 for cylinders we need the following:

Let G be a domain in $\mathbb{C}$. Let $f(x,y)$ be an analytic function of variables x,y, i.e. f is the restriction of a function $\tilde{f}(X,Y)$ holomorphic on a neighborhood of G in $\mathbb{C}^2$

to G. Following P. Davis [Da 2, Ch. 11], we shall say that *the function* f(x,y) *belongs to class* V(G) *(Vekua class), if the function* $\tilde{f}$ *is holomorphic in the domain* D *in* $\mathbb{C}^2$, *which is the image of* $G \times G$ *under the map*

$$(Z,W) \to \left\{\frac{Z + \overline{W}}{2}, \frac{Z - \overline{W}}{2i}\right\}.$$

In other words

$$D = \left\{(X,Y) : X = \frac{Z + \overline{W}}{2}, \ Y = \frac{Z - \overline{W}}{2i} ; \ Z,W \in G\right\}$$

(cf. [Hen], [Ve, Ch. 1], [KS 1]).

From the point of view of PDE, the significance of the Vekua class is this: through each point (X,Y) of $\mathbb{C}^2$ we can consider the two complex lines L = L(X,Y) and M = M(X,Y) defined by

$$L = \{(X + t, Y + it) : t \in \mathbb{C}\}$$

and

$$M = \{(X + t, Y - it) : t \in \mathbb{C}\}$$

These are the bicharacteristic lines through (X,Y) with respect to the Laplace operator (conceived as the operator $D_1^2 + D_2^2$ acting on holomorphic functions of two complex variables).

Each of these lines has a unique intersection with $\mathbb{R}^2 \cong \mathbb{C}$, (cf. [St]). For instance, L meets it in $\xi + i\eta$ (ξ,η real), where

$$X + t = \xi, \ Y + it = \eta$$

so that

(α) $\xi + i\eta = X + iY$

and, in like manner, M meets $\mathbb{R}^2 \cong \mathbb{C}$ in $a + ib$ (a,b real)

where

(β) $a - ib = X - iY$

From (α) and (β) we have

$$X = \frac{1}{2}[(\xi + i\eta) + (a - ib)]$$

$$Y = \frac{1}{2i}[(\xi + i\eta) - (a - ib)]$$

so that, writing $Z = \xi + i\eta$, $W = a + ib$

we have

$$X = \frac{1}{2}(Z + \overline{W}), \quad Y = \frac{1}{2i}(Z - \overline{W})$$

From these formulas we see that each point $(X,Y) \in \mathbb{C}^2$ is uniquely determined by the pair of points Z,W where the lines L(X,Y), M(X,Y) respectively meet $\mathbb{R}^2 \cong \mathbb{C}$. The "Vekua hull" of the plane domain G can be defined as the set of points (X,Y) in $\mathbb{C}^2$ such that the corresponding Z and W both lie in G, i.e. such that one can "see" G along each of the bicharacteristic lines through (X,Y). The Vekua class V(G) is then the set of functions holomorphically extendible from G to its Vekua hull.

Observe that taking $Z = W$, implies that $D \supset G$, and also, if $G = \mathbb{C}$, $D = \mathbb{C}^2$. The latter fact immediately shows that not all functions f analytic in $(x,y) \in G$, which are sometimes called in the literature real analytic in G, belong to $V(G)$. Indeed, the function $f(x,y) = \frac{1}{x - i} + \frac{1}{y - i}$ is holomorphic on a neighborhood in $\mathbb{C}^2$ of the whole complex plane $\mathbb{C} \equiv \mathbb{R}^2$, but its extension $\tilde{f}(X,Y)$ to $\mathbb{C}^2$ has singularities along two complex planes $\{X = i\}$ and $\{Y = i\}$.

On the other hand, the following example shows that the Vekua class always contains a vast variety of functions.

Example 3.10. (cf. [Da 2, p.123], [Hen, p. 3]. Let $u(x,y)$ be a harmonic function in a simply connected region $G \subset \mathbb{C}$. As is well known, there exist functions f and g, such that f is analytic in G, g is analytic in G^* where $G^* \stackrel{\text{def}}{=} \{\bar{z} : z \in G\}$ and

$$u(x,y) = u(z) = f(z) + g(\bar{z}) = f(x + iy) + g(x - iy), \quad z = x + iy \in G. \tag{3.6}$$

Now it is clear that f can be extended to a holomorphic function $\tilde{f}(X,Y)$, $\tilde{f}(X,Y) = f(X + iY)$ defined in a cylindrical domain $\hat{G} = \{(X,Y) \in \mathbb{C}^2 : Z = X + iY \in G\}$. Similarly g can be extended to a holomorphic function $\tilde{g}(X,Y) = g(X - iY)$ defined in $\hat{G}^* = \{(X,Y) \in \mathbb{C}^2 : \overline{W} = X - iY \in G^*\}$. Therefore, from

(3.6) it follows that the function

$$\tilde{u}(X,Y) = \tilde{u}\left((Z + \overline{W})/2,\ (Z - \overline{W})/2i\right) = f\ (Z) + g(\overline{W})$$

is holomorphic for $(Z,\overline{W}) \in G \times G^*$. Hence, $u \in V(G)$. This example admits the following important generalization, which will be used below (cf. [ACL, Ch.V]).

Let $v = v(x,y) = v(z)$ be a solution of the biharmonic equation $\Delta^2 v = 0$ in a simply connected region G. The formal integration of $\Delta^2 v = 16\ \partial^4 v/\partial z^2 \partial \overline{z}^2 = 0$ leads to the representation

$$v(x,y) = v(z) = \overline{z}f(z) + g(z) + zh(\overline{z}) + k(\overline{z}),$$

where f,g are analytic in G and h,k are analytic in G^*. Then, the above argument yields that the function

$$\tilde{v}(X,Y) = \tilde{v}\left((Z + \overline{W})/2\ ,\ (Z - \overline{W})/2i\right) = \overline{W}f(Z) + g(Z) + Zh(\overline{W}) + k(\overline{W})$$

is holomorphic for $(Z,\overline{W}) \in G \times G^*$. Thus, v also belongs to $V(G)$.

Applying a simple inductive procedure we obtain the following assertion:

Let $v(x,y)$ *be a solution of a polyharmonic equation* $\Delta^m v = 0$ *in a simply connected region* G. *Then,* $v \in V(G)$.

The following lemma sharpens the result in [Da2, p. 124].

<u>*Lemma 3.11*</u> *Let* $\Gamma \subset \mathbb{C}$ *be an analytic arc, and let* Ω *be a simply connected domain containing* Γ. *Let* $f(x,y) \in V(\Omega)$ *and assume that on* Γ $f(x,y) = P(x,y)$, *where* $P(x,y)$ *is a polynomial. Let* $v(x,y)$ *denote the solution of the Cauchy*

problem

$$\begin{cases} \Delta v = f \text{ near } \Gamma \\ v \equiv 0 \text{ on } \Gamma \end{cases}$$

Then, $v \in V(\Omega)$ *provided that the Schwarz function* S_Γ *is analytic throughout* Ω.

Remark. In particular, the conclusion of the lemma holds, when $P = 0$.

Proof. Let us switch to complex notation. So, the function

$F(Z,\overline{W}) = \tilde{f}(X,Y)$ where $X = \frac{Z + \overline{W}}{2}$, $Y = \frac{Z - \overline{W}}{2i}$ is holomorphic in $\Omega \times \Omega^*$ and $F(Z,\overline{Z}) = P(Z,\overline{Z}) = P\left(\frac{Z + \overline{Z}}{2}, \frac{Z - \overline{Z}}{2i} \right)$ on Γ.

Then, the initial value problem can be rewritten in the form ($W = \overline{Z}$ in Ω!)

$$\begin{cases} \dfrac{\partial^2}{\partial Z \partial \overline{Z}} v = \frac{1}{4} F(Z,\overline{Z}) \text{ near } \Gamma \\ v \equiv 0 \text{ on } \Gamma \end{cases} \qquad (3.7).$$

Fix $Z_0 \in \Gamma$ and define

$$v_0(Z,\overline{W}) = \frac{1}{4} \int_{\overline{Z}_0}^{\overline{W}} \int_{Z_0}^{Z} F(s,\overline{t}) ds d\overline{t},$$

where $(Z,\overline{W}) \in \Omega \times \Omega^*$ and also $(s,\overline{t}) \in \Omega \times \Omega^*$ for all s,t. Observe, that v_0 is well-defined, Indeed, since F is holomorphic in $\Omega \times \Omega^*$, the above integral does not depend on the path of integration. Moreover, it is clear that

$\tilde{v}_0(x,y) = v_0\left(\frac{Z+\overline{Z}}{2}, \frac{Z-\overline{Z}}{2i}\right)$ belongs to $V(\Omega)$ and

$\frac{\partial^2 v_0}{\partial Z \partial \overline{W}} = \frac{1}{4} F(Z,\overline{W})$. In particular, since $W = Z$ in Ω, it follows

that $\frac{\partial^2 v_0}{\partial Z \partial \overline{Z}} = \frac{1}{4} F(Z,\overline{Z})$. Set $u = v_0 - v$. Then, in view of (3.7), u is harmonic near Γ. Furthermore, since $v \equiv 0$ on Γ, we obtain

$$\left.\frac{\partial u}{\partial Z}\right|_\Gamma = \frac{1}{4}\int_{\overline{Z}_0}^{\overline{Z}} F(Z,\overline{t})d\overline{t} = \frac{1}{4}\int_{\overline{Z}0}^{\overline{Z}} P(Z,\overline{t})d\overline{t} = Q_1(Z,\overline{Z}) = Q_1(Z,S(Z)),$$

where $Q_1(Z,W)$ is a polynomial $(\frac{\partial Q_1}{\partial W}(Z,W) = P(Z,W))$.
Similiarly,

$$\left.\frac{\partial u}{\partial \overline{Z}}\right|_\Gamma = \frac{1}{4}\int_{Z_0}^{Z} P(s,\overline{Z})ds = Q_2(\overline{S(Z)},\overline{Z}),$$

where $Q_2(Z,W)$ is also a polynomial such that $\frac{\partial Q_2}{\partial Z}(Z,W) = P(Z,W)$. Thus, as in the proof of Theorem 3.1, the analytic function $\frac{\partial u}{\partial Z}$ and the antianalytic function $\frac{\partial u}{\partial \overline{Z}}$ coincide with $Q_1(Z,S(Z))$ and $Q_2 \ (\overline{S(Z)},\overline{Z})$ respectively, and are therefore analytically continuable throughout Ω. This, of course, implies that u is analytically extendible throughout Ω. As v_0, $u \in V(\Omega)$, so does v. The proof is complete.

Proposition 3.12. *Let* $\Gamma = \{(x_1,x_2,x_3)\colon (x_1,x_2) \in \gamma \subset \mathbb{R}^2\} = \gamma \times \mathbb{R}$

be a cylinder whose base γ *is an analytic arc in* $\mathbb{C}$. *Let* S_γ *be the Schwarz function of* γ *which is analytic in a simply connected domain* Ω. *Then, for any polynomial* $P(x_1,x_2,x_3)$ *the solution* u *of the initial value problem, (3.3) on* Γ *can be analytically extended into the solid cylinder* $\Omega \times \mathbb{R}$.

Proof. It suffices to verify our statement for polynomials $P=x_3^m P(x_1,x_2)$, $m=0,1,\ldots$, where $P(x_1,x_2)$ is a polynomial in x_1,x_2. For $m = 0,1$ take $u(x_1,x_2,x_3) = x_3^m u_0(x_1,x_2)$, $m = 0,1$, where u_0 is the solution of the Cauchy problem (3.1) on γ with Cauchy data given by $P(x_1,x_2)$. According to Theorem 3.1, u_0 can be analytically continued into Ω and the statement holds. Let $m = 2$.

Set

$$V_1(x_1,x_2,x_3) = u(x_1,x_2,x_3) - x_3^2\, u_0(x_1,x_2),$$

where u is a solution of (3.3) with the data $x_3^2 P(x_1,x_2) \equiv x_3^2 u_0(x_1,x_2)$.

Then, as in the calculation of the Schwarz potential of Γ, exploiting the invariance of Γ under the action of translations along the x_3-axis, we obtain for all $t \in \mathbb{R}$

$$u_t(x_1,x_2,x_3) \overset{\text{def}}{=} u(x_1,x_2,x_3 + t) - 2x_3 t u_0(x_1,x_2) - t^2 u_0(x_1,x_2) =$$

$$= x_3^2 u_0(x_1,x_2) + V_1(x_1,x_2,x_3 + t)$$

is also a solution of (3.3) with the data $x_3^2P(x_1,x_2)$. Hence, $u_t = u$. Therefore, for a fixed x_3 taking $t = -x_3$ we obtain

$$V_1(x_1,x_2,x_3) = V_1(x_1,x_2,0) = u(x_1,x_2,0).$$

So, $V_1(x_1,x_2,0)$ is a solution of the following initial value problem on γ

$$\begin{cases} V_1 \equiv 0 \text{ on } \gamma \\ \Delta V_1 = -2u_0(x_1,x_2). \end{cases} \tag{3.8}$$

Since u_0 is harmonic in Ω, $u_0 \in V(\Omega)$. Also, $u_0 = P(x_1,x_2)$ on γ. Thus V_1 satisfies the hypothesis of Lemma 3.11 and, therefore, V_1 is analytically continuable throughout Ω. So u is real-analytic in $\Omega \times \mathbb{R}$.

Let u_m denote the solution of the problem (3.3) with the initial data $x_3^mP(x_1,x_2)$. We will finish the proof of our proposition by induction. Thus, $u_0=u_0(x_1,x_2)$, $u_1=x_3u_0(x_1,x_2)$, $u_2=x_3^2u_0(x_1,x_2) + V_1$, where V_1 is a solution of (3.8).

Observe that

$$\frac{\partial u_0}{\partial x_3}(x_1,x_2,0) = 0, \ \frac{\partial u_1}{\partial x_3}(x_1,x_2,0) = u_0(x_1,x_2), \ u_1(x_1,x_2,0) = 0,$$

$$\frac{\partial u_2}{\partial x_3}(x_1,x_2,0) = 2u_1(x_1,x_2,0) = 0.$$

Assume that the proposition holds for $m = 0,1\ldots,2k$. Also assume that $u_0(x_1,x_2,0)$, $u_2(x_1,x_2,0)$, $\ldots$, $u_{2k}(x_1,x_2,0)$ are polyharmonic in Ω , $u_{2k}(x_1,x_2,0) \equiv 0$ on γ and

$$u_1(x_1,x_2,0) = u_3(x_1,x_2,0) = \dots = u_{2k-1}(x_1,x_2,0) = 0.$$

Once again, we write $u_{m+1}(x_1,x_2,x_3)$ as

$$u_{m+1} = x_3^{m+1}u_0(x_1,x_2) + V(x_1,x_2,x_3),$$

where $V \equiv 0$ on Γ. For each $t \in \mathbb{R}$, we have

$$u_{m+1}(x_1,x_2,x_3 + t) = x_3^{m+1}u_0(x_1,x_2) + V(x_1,x_2,x_3 + t) +$$

$$+ \sum_{i=1}^{m+1} \binom{m+1}{i} t^i x_3^{m+1-i}u_0(x_1,x_2).$$

Let

$$u_{m+1}^t(x_1,x_2,x_3) = u_{m+1}(x_1,x_2,x_3 + t) - \sum_{i=1}^{m+1}\binom{m+1}{i}t^i u_{m+1-i}.$$

Then, $\Delta u_{m+1}^t = 0$ near Γ, and on Γ we have

$$u_{m+1}^t\big|_\Gamma \equiv x_3^{m+1}u_0(x_1,x_2) + V(x_1,x_2,x_3 + t) \equiv x_3^{m+1}u_0(x_1,x_2).$$

Therefore, from the uniqueness of the solution of the Cauchy problem we conclude that $u_{m+1}^t = u_{m+1}$ and, again, letting $t = -x_3$ for a fixed x_3 we obtain

$$u_{m+1}(x_1,x_2,x_3) = u_{m+1}(x_1,x_2,0) - \sum_{i=1}^{m+1}(-1)^i\binom{m+1}{i}x_3^i u_{m+1-i}. \qquad (3.9)$$

From (3.9), it follows that

$$\left.\frac{\partial u_{m+1}}{\partial x_3}\right|_{x_3=0} = (m+1)u_m(x_1,x_2,0). \qquad (3.10)$$

Note that on γ, according to the definition of u_i, we have

$$u_{m+1}(x_1,x_2,0)\Big|_\gamma \equiv u_{m+1}(x_1,x_2,x_3)\Big|_\gamma + \sum_{i=1}^{m+1}(-1)^i\binom{m+1}{i}x_3^i u_{m+1-i} \equiv$$

$$(3.11)$$

$$\equiv x_3^{m+1}u_0(x_1,x_2) + \left[\sum_{i=0}^{m+1}(-1)^i(1)^{m+1-i}\binom{m+1}{i} -1\right]x_3^{m+1}u_0(x_1,x_2) \equiv 0.$$

As u_i are harmonic near Γ, we obtain from (3.9)

$$\Delta_2\, u_{m+1}(x_1,x_2,x_3) = \sum_{i=2}^{m+1} (-1)^i\binom{m+1}{i}i(i-1)x_3^{i-2}u_{m+1-i} +$$

$$(3.12)$$

$$+ 2\sum_{i=1}^{m+1} (-1)^i\binom{m+1}{i}ix_3^{i-1}\frac{\partial u_{m+1-i}}{\partial x_3}.$$

(Here, $\Delta_2=\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$). Hence,

$$\Delta_2 u_{m+1}(x_1,x_2,0) = m(m+1)u_{m-1}(x_1,x_2,0) - 2(m+1)\frac{\partial u_m}{\partial x_3}(x_1,x_2,0) =$$

$$(3.13)$$

$$= -m(m+1)u_{m-1}(x_1,x_2,0).$$

So, for $m = 2k$ we obtain, using our induction hypothesis, $\Delta_2 u_{m+1}(x_1,x_2,0) = 0$ near γ. This, in view of boundary condition (3.11), implies $u_{m+1}(x_1,x_2,0) = 0$. For $m = 2k+1$, (3.11), (3.13) imply that $u_{m+1}(x_1,x_2,0)$ is a solution of the initial value problem

$$\begin{cases} u_{m+1}(x_1,x_2,0) \equiv 0 \text{ on } \gamma \\ \Delta_2 u_{m+1}(x_1,x_2,0) = -m(m+1)u_{m-1}(x_1,x_2,0). \end{cases}$$

Since by the induction hypothesis $u_{m-1}(x_1,x_2,0) = u_{2k}(x_1,x_2,0)$ vanishes on γ and is polyharmonic in Ω, then repeating the argument we have given in the solution of problem (3.8) and using Lemma 3.11 again, we conclude that $u_{m+1}(x_1,x_2,0)$ is real analytic throughout Ω. Thus, from (3.9) it follows that both u_{2k+1}, u_{2k+2} are analytically continuable to $\Omega \times \mathbb{R}$. This finishes the verification of the induction step and the proof of our proposition is now complete.

Corollary 3.13. Let Γ *be the circular cylinder in* $\mathbb{R}^3$ *defined as* $\{(x_1,x_2,x_3)\colon x_1^2+x_2^2 = R^2\}$. *Then, for any polynomial* $P(x_1,x_2,x_3)$ *the solution of the initial value problem (3.3) with the Cauchy data* P *on* Γ, *can be analytically continued to* $\mathbb{R}^3\backslash\{(0,0,x_3), x_3 \in \mathbb{R}\}$.

Proof. Observe that for the circle $x_1^2+x_2^2 = R^2$, the Schwarz function $S(z) = R^2/z$. So, $\Omega = \mathbb{C}\backslash\{0\}$ and we can apply

Proposition 3.12.

Remark. We have not been able to obtain an analog of Proposition 3.12 for cylinders of higher dimensions since our method would require to verify Conjecture 3.4 in $\mathbb{R}^3$ first. However, Corollary 3.13 can be extended to dimensions higher than 3.

Corollary 3.14. Let Γ *be a spherical cylinder in* $\mathbb{R}^n$*, i.e.*

$$\Gamma = \{(x_1,\dots,x_n): \sum_{i=1}^{n-1} x_i^2 = R^2\}.$$

Then, for any polynomial $P(x_1,\dots,x_n)$ *the solution of the initial value problem (3.3) can be continued as a real-analytic function to the complement of the axis of the cylinder.*

Proof. Without loss of generality, we can assume that $R = 1$. Again it suffices to solve the problems

$$\begin{cases} u_m \equiv x_n^m P(x_1,\dots,x_{n-1}) \text{ on } \Gamma \\ \Delta u_m = 0 \text{ near } \Gamma \end{cases}$$

$m = 0,1,\dots$. For $m = 0$, put $u_0 = H(x_1,\dots,x_{n-1})$ where H is a solution of the problem

$$\begin{cases} H \equiv P(x_1,\dots,x_{n-1}) \text{on } S^{n-2} \\ \Delta H = 0 \text{ near } S^{n-2} \end{cases} \tag{3.14}$$

where $S^{n-2} = \{(x_1,\dots,x_{n-1}): \sum_{i=1}^{n-1} x_i^2 = 1\}$. Since Proposition 3.7, as is readily seen from the proof, holds for higher dimensions, u_0 is analytic in $\mathbb{R}^{n-1}\setminus\{0\}$. For $m = 1$, take $u_1 = x_n H$, where H is the solution of (3.14). For $m = 2$, as in the proof of Proposition 3.12, it suffices to solve the problem

$$\begin{cases} u_2(x_1,\dots,x_{n-1},0) \equiv 0 \text{ on } S^{n-2} \\ \Delta_{n-1}u_2(x_1,\dots,x_{n-1},0) = -2H \text{ near } S^{n-2} \end{cases} \tag{3.15}$$

From the proof of Proposition 3.7, it follows that $H(x_1,\dots,x_{n-1})$ is a linear combination of H_k, and $\dfrac{H_k}{|x|_{n-1}^{2k+n-3}}$, where $H_k = H_k(x_1,\dots,x_{n-1})$ are homogeneous harmonic polynomials of degree k. Thus, to solve (3.15) it suffices to solve

$$\text{(i)} \begin{cases} u \equiv 0 \text{ on } S^{n-2} \\ \Delta u = H_k \text{ near } S^{n-2} \end{cases} \quad \text{(ii)} \begin{cases} u \equiv 0 \text{ on } S^{n-2} \\ \Delta u = \dfrac{H_k}{|x|_{n-1}^{2k+n-3}} \end{cases}.$$

(i) From Lemma 3.8 it follows that there exist a polynomial $P(x_1,\dots,x_{n-1}) \in P_{k+2}$: $\Delta P = H_k$ and a harmonic polynomial $H_{k+2} \in P_{k+2}$ such that $H_{k+2} = P$ on S^{n-2}. Also, as it follows from (3.4), there is a harmonic polynomial H_k' in P_k such that

$$P - H_{k+2} + (1 - \sum_{i=1}^{n-1} x_i^2)H_k' = 0 \text{ in } \mathbb{R}^{n-1}.$$

Let $H_k' = \sum_{i=0}^{k} h_i$, where $h_i \in P_i$ are homogeneous harmonic polynomials.

As for $i = 0,\ldots,k$

$$\frac{2}{2i+n-3}\left(h_i - \frac{h_i}{|x|_{n-1}^{2i+n-3}}\right) \equiv \begin{pmatrix} 0 \\ 2x_1h_i,\ldots,2x_{n-1}h_i \end{pmatrix} \equiv$$

$$\equiv \left(\sum_{\ell=1}^{n-1} x_\ell^2 - 1\right)h_i$$

on S^{n-2}, (see the end of the proof of Proposition 3.7), we have

$$\sum_{i=0}^{k} \frac{2}{2i+n-3}\left(h_i - \frac{h_i}{|x|_{n-1}^{2i+n-3}}\right) \equiv (|x|_{n-1}^2 - 1)H_k' \text{ on } S^{n-2}.$$

Hence,

$$u = P - H_{k+2} - \sum_{i=0}^{k} \frac{2}{2i+n-3}\left(h_i - \frac{h_i}{|x|_{n-1}^{2i+n-3}}\right)$$

solves (i).

For (ii), observe that since H_k is homogeneous, a direct calculation yields

$$-\frac{1}{2(2k+n-5)}\,\Delta\left(\frac{H_k}{|x|_{n-1}^{2k+n-5}}\right) = \frac{H_k}{|x|_{n-1}^{2k+n-3}}$$

Also,

$$\frac{H_k}{|x|_{n-1}^{2k+n-5}}\Bigg|_{S^{n-2}} \equiv \begin{pmatrix} H_k \\ \nabla H_k \end{pmatrix} - (2k+n-5)\begin{pmatrix} 0 \\ x_1H_k,\ldots,x_{n-1}H_k \end{pmatrix}.$$

Therefore, as above, we obtain

$$\frac{H_k}{|x|_{n-1}^{2k+n-5}} \equiv H_k - \frac{2k+n-5}{2k+n-3}\left(H_k - \frac{H_k}{|x|_{n-1}^{2k+n-3}}\right) \equiv$$

$$\equiv \frac{2}{2k+n-3} H_k + \frac{2k+n-5}{2k+n-3} \frac{H_k}{|x|_{n-1}^{2k+n-3}}$$

on S^{n-2}. Thus, from Lemma 3.9 it follows that

$$u = -\frac{1}{2(2k+n-5)} \frac{H_k}{|x|_{n-1}^{2k+n-5}} + \frac{H_k}{(2k+n-5)(2k+n-3)}$$

$$+ \frac{1}{2(2k+n-3)} \frac{H_k}{|x|_{n-1}^{2k+n-3}}$$

solves (ii). Hence, the solutions of (i) and (ii), and therefore the solutions of (3.15) are regular in $\mathbb{R}^{n-1}\backslash\{0\}$. Repeating the same type of calculations one can readily verify that the solution of the problem

$$\begin{cases} u_m(x_1,\ldots,x_{n-1},0) \equiv 0 \text{ on } S^{n-2} \\ \Delta u_m = -m(m+1)u_{m-2}(x_1,\ldots,x_{n-1},0) \text{ near } S^{n-2} \end{cases}$$

for all even m is expressible as a finite linear combination of polynomials and functions of the form $\frac{H_k}{|x|_{n-1}^{2\ell+n-3}}$, $\ell \leq k$, and, therefore, is regular in $\mathbb{R}^{n-1}\backslash\{0\}$. Then, repeating the same induction step as in the proof of Proposition 3.12 we complete the proof of our Corollary.

Remark 3.15. Johnsson [Jo 2] has also generalized the above Corollary to the case of arbitrary Cauchy data holomorphic in a neighborhood of the "complexified" cylinder $\hat{\Gamma} = \{z \in \mathbb{C}^n : \sum_1^{n-1} z_i^2 = 1\}$. In particular, Corollary 3.14 holds for entire Cauchy data.

Finally, let us discuss Conjecture 3.4 for the cone $\Gamma: x_1^2 + x_2^2 = x_3^2$.

<u>*Proposition 3.16*</u>. *Let* Γ *denote the right circular cone* $x_1^2 + x_2^2 = x_3^2$, $x_3 > 0$; *then Conjecture 3.4 holds for arbitrary polynomial data on* Γ, *i.e. the solution of initial value problem (3.3) for an arbitrary polynomial* $P(x_1,x_2,x_3) = P(x)$ *can be analytically continued to* $\Omega = \{(x_1,x_2,x_3)\colon x_3 > 0,\ x_1^2 + x_2^2 > 0\}$.

<u>*Proof*</u>. Without loss of generality we can assume that P is a homogeneous polynomial of degree n. Let u be a solution of our problem near Γ. Then, for arbitrary $t > 0$ we have $(x=(x_1,x_2,x_3))$

$$u(tx) \equiv P(tx) \equiv t^n P(x) \text{ on } \Gamma.$$

So,

$$\frac{u(tx)}{t^n} \equiv u(x) \text{ on } \Gamma.$$

Since, $\frac{1}{t^n} u(tx)$ is harmonic near Γ for fixed t, from the uniqueness of the solution of the Cauchy problem we conclude that

$$\frac{1}{t^n} u(tx) = u(x) \text{ near } \Gamma.$$

For fixed x_1, x_2, x_3 taking $t = \frac{1}{x_3}$ we obtain

$$u(x) = x_3^n \, u(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1).$$

Introducing new variables $s = x_1/x_3$, $t = x_2/x_3, x_3$ and setting $v(s,t) = u(x_1/x_3, x_2/x_3, 1)$, we obtain

$$u(x_1,x_2,x_3) = x_3^n \, v(s,t) \text{ near } \Gamma.$$

Differentiating, we find

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x_1} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x_1} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial x_1} = x_3^{n-1}\frac{\partial v}{\partial s},$$

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial}{\partial s}(x_3^{n-1}\frac{\partial v}{\partial s})\frac{\partial s}{\partial x_1} = x_3^{n-2}\frac{\partial^2 v}{\partial s^2}. \tag{3.16}$$

Similarly,

$$\frac{\partial^2 u}{\partial x_2^2} = x_3^{n-2}\frac{\partial^2 v}{\partial t^2}. \tag{3.17}$$

Also, setting

$$\phi(s,t,x_3) \overset{\text{def}}{=} \frac{\partial u}{\partial x_3} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x_3} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x_3} + \frac{\partial u}{\partial x_3} = x_3^n \frac{\partial v}{\partial s}(-\frac{x_1}{x_3^2}) +$$

$$+ x_3^n \frac{\partial v}{\partial t} (-\frac{x_2}{x_3^2}) + n x_3^{n-1} v(s,t) = n x_3^{n-1} v - x_3^{n-2} \frac{\partial v}{\partial s} x_1 -$$

$$- x_3^{n-2} \frac{\partial v}{\partial t} x_2,$$

we find

$$\frac{\partial^2 u}{\partial x_3^2} = \frac{\partial \phi}{\partial x_3} + \frac{\partial \phi}{\partial s} (-\frac{x_1}{x_3^2}) + \frac{\partial \phi}{\partial t}(-\frac{x_2}{x_3^2}) =$$

$$= n(n-1) x_3^{n-2} v - (n-1) x_3^{n-2} \frac{\partial v}{\partial s} s - (n-1) x_3^{n-2} \frac{\partial v}{\partial t} t -$$

$$- \frac{s}{x_3} \left(n x_3^{n-1} \frac{\partial v}{\partial s} - x_3^{n-1} \frac{\partial v}{\partial s} - x_3^{n-1} s \frac{\partial^2 v}{\partial s^2} - x_3^{n-1} t \frac{\partial^2 v}{\partial t \partial s} \right)$$

$$- \frac{t}{x_3} \left(n x_3^{n-1} \frac{\partial v}{\partial t} - s x_3^{n-1} \frac{\partial^2 v}{\partial s \partial t} - x_3^{n-1} \frac{\partial v}{\partial t} - x_3^{n-1} t \frac{\partial^2 v}{\partial t^2} \right)$$

$$= n(n-1) x_3^{n-2} v - 2(n-1) x_3^{n-2} (\frac{\partial v}{\partial s} s + \frac{\partial v}{\partial t} t) + x_3^{n-2} (s^2 \frac{\partial^2 v}{\partial s^2} +$$

$$+ t^2 \frac{\partial^2 v}{\partial t^2}) + 2st\, x_3^{n-2} \frac{\partial^2 v}{\partial s \partial t}. \qquad (3.18)$$

Thus, adding (3.16), (3.17), and (3.18), recalling that $\Delta u = 0$ near Γ and dividing by x_3^{n-2} we obtain an elliptic equation for $v(s,t)$:

$$(1+s^2)\frac{\partial^2 v}{\partial s^2} + (1+t^2)\frac{\partial^2 v}{\partial t^2} + 2st\,\frac{\partial^2 v}{\partial s\partial t} - 2(n-1)[\frac{\partial v}{\partial s}s + \frac{\partial v}{\partial t}t] + \qquad (3.19)$$

$$+ n(n-1)\ v = 0$$

holding near $s^2 + t^2 = 1$. Furthermore, assuming without loss of generality that our polynomial P is a monomial $x_3^k x_1^\ell x_2^m$, $k + \ell + m = n$, it is easy to check that on the unit circle γ: $s^2 + t^2 = 1$ we have (see (3.16))

$$v(s,t) = \frac{1}{x_3^n}(x_3^k x_1^\ell x_2^m) = s^\ell t^m = q(s,t)$$

$$\frac{\partial v}{\partial s} = \frac{\frac{\partial u}{\partial x_1}}{x_3^{n-1}} = \ell s^{\ell-1} t^m = \frac{\partial q}{\partial s}$$

$$\frac{\partial v}{\partial t} = \frac{\partial q}{\partial t}.$$

Hence, on γ, $v \equiv q(s,t)$. Switching to polar coordinates $s = r\cos\theta$, $t = r\sin\theta$, $r = \sqrt{s^2 + t^2}$ we find

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\sin\theta\,\frac{\partial v}{\partial \theta};$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\cos\theta\,\frac{\partial v}{\partial \theta};$$

$$\frac{\partial^2 v}{\partial s^2} = \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\sin\theta\,\frac{\partial v}{\partial \theta}\right) - \frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\cos\theta\right.$$

$$- \frac{1}{r} \sin \theta \frac{\partial v}{\partial \theta}) = \cos^2\theta \frac{\partial^2 v}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial v}{\partial \theta} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 v}{\partial r \partial \theta}$$

$$+ \frac{\sin^2\theta}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \sin \theta \cos \theta \frac{\partial v}{\partial \theta} + \frac{1}{r^2} \sin^2\theta \frac{\partial^2 v}{\partial \theta^2};$$

$$\frac{\partial^2 v}{\partial t^2} = \sin^2\theta \frac{\partial^2 v}{\partial r^2} - \frac{2}{r^2} \cos \theta \sin \theta \frac{\partial v}{\partial \theta} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 v}{\partial r \partial \theta} +$$

$$\frac{1}{r}\cos^2\theta\frac{\partial v}{\partial r} + \frac{1}{r^2} \cos^2 \theta \frac{\partial^2 v}{\partial \theta^2};$$

$$\frac{\partial^2 v}{\partial s \partial t} = \sin\theta \frac{\partial}{\partial r}(\frac{\partial v}{\partial r} \cos \theta - \frac{1}{r} \sin \theta \frac{\partial v}{\partial \theta}) +$$

$$+ \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta}(\frac{\partial v}{\partial r} \cos \theta - \frac{1}{r} \sin \theta \frac{\partial v}{\partial \theta}) = \sin \theta \cos \theta \frac{\partial^2 v}{\partial r^2} +$$

$$+ \frac{\sin^2\theta}{r^2} \frac{\partial v}{\partial \theta} - \frac{\sin^2\theta}{r} \frac{\partial^2 v}{\partial r \partial \theta} + \frac{\cos^2\theta}{r} \frac{\partial^2 v}{\partial r \partial \theta} - \frac{\cos \theta \sin \theta}{r} \frac{\partial v}{\partial r} -$$

$$- \frac{\sin \theta \cos \theta}{r^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{\cos^2\theta}{r^2} \frac{\partial v}{\partial \theta}.$$

Thus, our equation can be written in polar coordinates as

$$(1 + r^2\cos^2\theta)\left(\cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial v}{\partial \theta} - \frac{2 \cos \theta \sin \theta}{r} \frac{\partial^2 v}{\partial r \partial \theta} +\right.$$

$$\left.\frac{\sin^2\theta}{r} \frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r^2} \frac{\partial^2 v}{\partial \theta^2}\right) + (1 + r^2\sin^2\theta)\left(\sin^2\theta \frac{\partial^2 v}{\partial r^2} - \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial v}{\partial \theta}\right.$$

$$+ \frac{2 \sin\theta \cos\theta}{r} \frac{\partial^2 v}{\partial r \partial\theta} + \frac{\cos^2\theta}{r} \frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r^2} \frac{\partial^2 v}{\partial\theta^2}\Bigg) +$$

$$+ 2r^2 \cos\theta \sin\theta \Bigg(\sin\theta \cos\theta \frac{\partial^2 v}{\partial r^2} + \frac{\sin^2\theta}{r^2} \frac{\partial v}{\partial\theta} - \frac{\sin^2\theta}{r} \frac{\partial^2 v}{\partial r \partial\theta} +$$

$$+ \frac{\cos^2\theta}{r} \frac{\partial^2 v}{\partial r \partial\theta} - \frac{\cos\theta \sin\theta}{r} \frac{\partial v}{\partial r} - \frac{\sin\theta \cos\theta}{r^2} \frac{\partial^2 v}{\partial\theta^2} - \frac{\cos^2\theta}{r^2} \frac{\partial v}{\partial\theta}\Bigg)$$

$$- 2(n-1)\Bigg(r \cos^2\theta \frac{\partial v}{\partial r} - \sin\theta\cos\theta \frac{\partial v}{\partial\theta} + r \sin^2\theta \frac{\partial v}{\partial r} + \cos\theta\sin\theta \frac{\partial v}{\partial\theta}\Bigg)$$

$$+ n(n-1)\, v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial\theta^2} - 2(n-1) r \frac{\partial v}{\partial r} + r^2 \frac{\partial^2 v}{\partial r^2}$$

$$+ n(n-1)\, v = 0 \qquad (3.20)$$

near γ. Observe that on the unit circle γ: $r = 1$

$$v(r,\theta) = r^{\ell+m} \cos^{\ell}\theta \sin^{m}\theta = \cos^{\ell}\theta \sin^{m}\theta:$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial s}\cos\theta + \frac{\partial v}{\partial t}\sin\theta = (\ell+m) r^{m+\ell-1} \cos^{\ell}\theta \sin^{m}\theta = (\ell+m) \cos^{\ell}\theta \sin^{m}\theta:$$

$$\frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial s}(-r \sin\theta) + \frac{\partial v}{\partial t}(r \cos\theta) = -\ell r^{m+\ell} \cos^{\ell-1}\theta \sin^{m+1}\theta +$$

$$+ m r^{m+\ell} \cos^{\ell+1}\theta \sin^{m-1}\theta =$$

$$= -\ell \cos^{\ell-1}\theta \sin^{m+1}\theta + m \cos^{\ell+1}\theta \sin^{m-1}\theta.$$

Hence, the solution $v(r,\theta)$ of (3.20) satisfies on the unit circle the initial data

$$v(r,\theta) \equiv r^{\ell+m} \cos^{\ell}\theta \sin^{m}\theta.$$

Denote by $L: = (1 + r^2) \frac{\partial^2}{\partial r^2} + (\frac{1}{r} - 2(n-1)r)\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} +$ $+ n(n-1)$ the linear differential operator in (3.20). As $\cos^{\ell}\theta\sin^{m}\theta$ can be written in the form of a trigonometric polynomial $\sum_0^N(a_\nu\cos(\omega_\nu\theta) + b_\nu(\sin(\omega_\nu\theta))$, to complete the proof of our proposition it suffices to show that solutions of the initial value problems

$$\text{(i)} \begin{cases} w \equiv r^{\nu}\cos\omega\theta \text{ on } \gamma \\ Lw = 0 \text{ near } \gamma \end{cases} \qquad \text{(ii)} \begin{cases} w \equiv r^{\nu}\sin\omega\theta \text{ on } \gamma \\ Lw = 0 \text{ near } \gamma \end{cases}, \quad \nu=0,1,2,\ldots,$$

are analytically continuable to $\mathbb{R}^2\backslash\{0\}$. Since the problems (i) and (ii) are essentially identical we shall conduct the argument for the initial value problem (i). Separating the variables in (3.20) and writing $w(r,\theta) = \phi(r)\psi(\theta)$ we find that (3.20) yields

$$(r^2+r^4)\phi''\psi + (r - 2(n-1)r^3)\phi'\psi + n(n-1)r^2\phi\psi = -\phi\psi'',$$

or

$$(r^2+r^4)\frac{\phi''}{\phi} + (r - 2(n-1)r^3)\frac{\phi'}{\phi} + n(n-1)r^2 = -\frac{\psi''}{\psi}.$$

Hence, taking $\psi(\theta) = \cos\omega\theta$ we obtain that $w = \phi(r)\cos\omega\theta$ solves (i) if and only if ϕ is a solution of the ordinary differential equation

$$(r^2 + r^4)\phi'' + (r - 2(n-1)r^3)\phi' + [n(n-1)r^2 - \omega^2]\phi = 0$$

satisfying the initial data $\phi(1) = 1$ $\phi'(1) = \nu$.

From the standard theory of ODE (see [In], Ch. III, Ch. IV, Ch. XII) it follows that ϕ can be continued as a real-analytic function to the positive real axis. This finishes the proof of our proposition.

Remark 3.17. Let us sketch a different proof of Theorem 3.1. Having found the Schwarz potential of an analytic curve Γ, we have solved the initial value problem (3.1) for all polynomials of degree ≤ 2. Let u be the solution of the initial value problem

$$\begin{cases} u \equiv x^2 \text{ on } \Gamma \\ \Delta u = o \text{ near } \Gamma \end{cases}.$$

Hence, ∇u is an antianalytic function near Γ. If we identify the vector ∇u with a complex number $2\frac{\partial u}{\partial \bar{z}}$, then, $(\nabla u)^2 = 4(\frac{\partial u}{\partial \bar{z}})^2$ is again an antianalytic function. So, there is a harmonic function v such that $\nabla v = (\frac{\partial u}{\partial \bar{z}})^2$. But on Γ, $\nabla v = (x^2,0)$, so $v \equiv (1/3)x^3 + \text{const}$ on Γ. Also, it is clear that v can be analytically continued to the region of analyticity of $\frac{\partial u}{\partial \bar{z}}$ which coincides with the region of analyticity of u. Repeating this argument we obtain Theorem 3.1 for all cubics and then a simple induction step will complete the proof of

Theorem 3.1 in case of polynomial data. This argument fails in $\mathbb{R}^n$ $(n \geq 3)$ because no analogous multiplication for gradients of harmonic functions is available.

§4. The Schwarz potential and quadrature domains for harmonic functions.

The characteristic property of a harmonic function u is its mean value property

$$u(x_0) = \frac{1}{\mathrm{Vol}(B)} \int_B u\,dx. \tag{4.1}$$

Here, B stands for a ball centered at $x_0 \in \mathbb{R}^n$, Vol B denotes its volume, and dx denotes Lebesgue measure in $\mathbb{R}^n$. In fact, it is well known that, together with some continuity assumption on u, (4.1) is equivalent to the property of u being harmonic. At the same time, one can also view (4.1) as the simplest "quadrature identity": to integrate a harmonic function over a ball B it suffices to evaluate it at the center of B. Moreover, it has been proved by many authors with some minor differences in the hypotheses that balls in $\mathbb{R}^n$ are characterized by this identity (e.g. see [Ep], [ES], [Ku], [ASZ]).

The concept of a "quadrature domain" can be essentially generalized. For our purposes we shall accept the following definition (cf. [AS], [Da 2], Ch., XIV, [Sa 1], [Sh 1,2,4].

<u>*Definition 4.1*</u>. *A bounded connected open set* $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *is called a quadrature domain if there exists a distribution* T *compactly supported in* Ω *such that*

$$\int_\Omega u dx = \langle T,u\rangle \tag{4.2}$$

for all functions u *integrable on* Ω *and harmonic in* Ω *(i.e.* $u \in L^1_h(\Omega)$*).*

Let us briefly comment on the (by now considerable) history of such problems. The earliest nontrivial quadrature identity for analytic functions is, as mentioned in the introduction, implicit in [Ne] and the modern theory was initiated by Philip Davis (see [Da 1,2]); see also [AS], [Sh 4].

The "quadrature identities" studied there are concerned with analytic functions and related to distributions T of a special kind, namely, finite linear combinations of δ-functions and their derivatives taken at finitely many points in Ω. Further investigation of the quadrature domains of this type and quadrature domains of a more general nature in $\mathbb{R}^2$, with T being a Borel measure in Ω (not necessarily of compact support), has been done by B. Gustafsson [Gu 1,2] and M. Sakai [Sa 1,4]. For further information the reader may consult Sakai's book [Sa 1] and the references cited there. [Sh 1] is probably the first paper dealing with quadrature identities more general than (4.1) for harmonic functions in space. Unbounded quadrature domains have been studied in [Sh 2] and [Sha 2,3] , and will be discussed in §5, IV.

The following theorem readily generalizes the corresponding result in [Sh 1] (Theorem 3.1), where T was assumed to be a finite linear combination of point-masses. The proof we present here, however, is different from the one given in [Sh 1], and based on an elementary potential-theoretic argument.

First, let us introduce some notation. If T is a distribution in $\mathbb{R}^n$ with compact support, then we denote by $\hat{T}$ the convolution

$$\hat{T} = T * k_n,$$

where

$$k_n = \begin{cases} -\frac{1}{2\pi} \log \frac{1}{|x|} & n = 2 \\ -\frac{1}{\omega_n} \frac{1}{n-2} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

is the fundamental solution for the Laplace operator. (Here, ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$, $n \geq 3$). Since T has compact support and k_n is locally integrable in $\mathbb{R}^n$, $\hat{T}$ is a well-defined distribution on $\mathbb{R}^n$. Moreover it is obvious that $\hat{T}$ is harmonic outside of supp T. (cf. [Ru, Ch. VI]).

__Theorem 4.2__. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *be a bounded open set and suppose the quadrature identity (4.2) holds for all* $u \in L^1_h(\Omega)$. *Then, there exists a function* u_0 *harmonic in* Ω *whose gradient is continuously extendible to* Γ *such that*

$$\hat{T} - \frac{1}{2n}|x|^2 \equiv u_0 \tag{4.3}$$

on the boundary Γ *of* Ω.

Conversely, let $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set with a Jordan rectifiable boundary* Γ *(cf. [Fe, Ch. IV]). If there exists a function* $u_0 \in C^1(\bar{\Omega})$ *such that (4.3) holds on* Γ *for a distribution* T *compactly supported in* Ω, *then* Ω *is a quadrature domain and (4.2) holds.*

Proof. Set

$$\hat{\Omega}(x) = \chi_\Omega * k_n,$$

where χ_Ω is the characteristic function of Ω. Then, as is well known (see e.g. [Ke, Ch. VI]) $\hat{\Omega}(x) \in C^1(\mathbb{R}^n)$, $\hat{\Omega}(x)$ is harmonic outside of $\bar{\Omega}$ and $\Delta\hat{\Omega} = 1$ in Ω.

Since T is compactly supported in Ω, $k_n(x)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ and both $k_n(x)$ and its derivatives $\frac{\partial}{\partial x} k_n(x)$, $i = 1,\ldots,n$ are locally integrable we obtain from (4.2) that

$$\hat{\Omega}(x) = \hat{T}(x)$$

$$\frac{\partial}{\partial x_i} \hat{\Omega}(x) = \frac{\partial}{\partial x_i} \hat{T}(x), \ i = 1,\ldots,n$$

for all $x \in \mathbb{R}^n \setminus \Omega$. Therefore, in particular,

$$\hat{\Omega}(x) \equiv \hat{T}(x) \text{ on } \Gamma. \tag{4.4}$$

As $\Delta((1/2n)|x|^2) = 1$, the function

$$u_0(x) = \hat{\Omega}(x) - \frac{1}{2n}|x|^2 \qquad (4.5)$$

is harmonic in Ω and also $u_0 \in C^1(\mathbb{R}^n)$. Substituting (4.4) into (4.5) we obtain (4.3).

Conversely, let $\Gamma = \partial\Omega$ be rectifiable and (4.3) hold on Γ. At first, let us show that (4.2) is valid for u harmonic in $\bar{\Omega}$. To do this it suffices to verify (4.2) for $u(x) = k_n(x-y)$ for all $y \in \mathbb{R}^n\setminus\bar{\Omega}$. Since Ω has a finite perimeter the Green theorem holds for Ω. (See e.g. [Fe]).

As for a fixed $y \in \mathbb{R}^n\setminus\bar{\Omega}$, $k_n(x-y) \in C^\infty(\bar{\Omega})$, and harmonic in $\bar{\Omega}$, from (4.3) we obtain

$$\int_\Omega k_n(x-y)dx = \int_\Omega \left\{k_n(x-y)\Delta(\frac{1}{2n}|x|^2) - \Delta k_n(x-y)(\frac{1}{2n}|x|^2)\right\}dx$$

$$= \int_\Gamma \left\{k_n(x-y)\,\frac{\partial}{\partial n_x}\,(\hat{T} - u_0) - \frac{\partial k_n(x-y)}{\partial n_x}(\hat{T} - u_0)\right\}dS_x,$$

where dS_x is the Lebesgue measure on Γ and n_x is the outer normal to Γ at x (n_x exists a.e. on Γ since Γ is assumed to be rectifiable - cf. [Fe, Ch. IV]). As u_0 is harmonic in Ω, Green's second identity yields

$$\int_\Gamma \{k_n \frac{\partial u_0}{\partial n} - \frac{\partial k_n}{\partial n} u_0\}dS = 0.$$

Thus,

$$\int_\Omega k_n(x-y)dx = \int_\Gamma \left\{ k_n(x-y) \frac{\partial}{\partial n_x} \hat{T}(x) - \frac{\partial k_n(x-y)}{\partial n_x} \hat{T}(x) \right\} dS_x. \qquad (4.6)$$

It is convenient to single out the following assertion.

Assertion. Let $\phi \in C^\infty(\bar{\Omega})$. Then,

$$\int_\Gamma \left\{ \phi \frac{\partial \hat{T}}{\partial n} - \frac{\partial \phi}{\partial n} \hat{T} \right\} dS = \langle \phi, T \rangle - \langle \Delta\phi, \hat{T}\chi_\Omega \rangle. \qquad (4.7)$$

Let us assume (4.7) for the moment. Substituting $\phi(x) = k_n(x-y)$ from (4.6) and recalling that $\Delta k_n(x-y) = 0$ in $\bar{\Omega}$, we obtain from (4.7)

$$\int_\Omega k_n(x-y)dx = \langle k_n(x-y), T \rangle = \hat{T}(y). \qquad (4.8)$$

This is precisely what we had to show. Note, that if Γ is regular with respect to the Dirichlet problem, then (4.2) already follows from (4.8), since it is well-known that for such Ω, functions harmonic in a neighborhood of $\bar{\Omega}$ are dense in L^p_h, $p \geq 1$ (cf. e.g. [He 1,2]. However, for more general Γ, we have to verify (4.8) also for $y \in \Gamma$. Since the system of functions $\{k_n(x-y),\ y \in \mathbb{R}^n \backslash \Omega\}$ is complete in L^1_h (see [He 1,2], [Po]), this would imply that (4.2) holds for all $u \in L^1_h$. Fix $y_0 \in \Gamma$ and $\epsilon > 0$. Let Ω_ϵ: supp $T \subset \Omega_\epsilon \subset \Omega$ be a domain with a smooth rectifiable boundary such that for all $x \in \partial\Omega_\epsilon$, the following conditions are satisfied

(a) $\mathrm{dist}(x,\Gamma) < \epsilon$

$$\text{(b)}\quad \|\nabla(\hat{T} - \frac{1}{2n}|x|^2 - u_0)\|_{C(\partial\Omega_\epsilon)} \le \epsilon \tag{4.9}$$

(c) $\|\partial\Omega_\epsilon\| \overset{\text{def}}{=} \text{Perimeter of } \Omega_\epsilon \le \text{const} < +\infty$, independent of ϵ.

(The existence of Ω_ϵ satisfying (a), (c) easily follows from a general theory of sets with finite perimeter, cf. [Fe, Ch. IV]). Set $\delta(x) = \hat{T} - \frac{1}{2n}|x|^2 - u_0$. Then, using harmonicity of $u_0, k_n(x-y_0)$ in Ω_ϵ, Green's identity and (4.7) we find

$$\int_{\Omega_\epsilon} k_n(x-y_0)dx = \int_{\Omega_\epsilon} k_n(x-y_0)\Delta(\frac{1}{2n}|x|^2)dx =$$

$$= \int_{\partial\Omega_\epsilon} \left\{k_n(x-y_0)\frac{\partial}{\partial n_x}(\hat{T}(x) - u_0(x)) - \frac{\partial k_n(x-y_0)}{\partial n_x}(\hat{T}(x) - u_0(x)\right\}dS_x -$$

$$- \int_{\partial\Omega_\epsilon} \left\{k_n(x-y_0)\frac{\partial}{\partial n_x}\,\delta(x) - \frac{\partial k_n(x-y_0)}{\partial n_x}\,\delta(x)\right\}dS_x$$

$$= \hat{T}(y_0) - \int_{\partial\Omega_\epsilon} \left\{k_n(x-y_0)\frac{\partial\delta(x)}{\partial n_x} - \delta(x)\frac{\partial k_n(x-y_0)}{\partial n_x}\right\}dS_x. \tag{4.10}$$

From (4.9(b)) it readily follows that

$$\left|\int_{\partial\Omega_\epsilon} k_n(x-y_0)\frac{\partial\delta(x)}{\partial n_x}dS_x\right| \le \epsilon \int_{\partial\Omega_\epsilon} |k_n(x-y_0)|dS_x \le \epsilon \text{ const}\|\partial\Omega_\epsilon\|. \quad (4.11)$$

Also, for each $x \in \Omega_\epsilon$, $|\frac{\partial}{\partial n} k_n(x-y_0)| \le \text{const}|x-y_0|^{1-n}$. On the other hand, if $x \in \Gamma$ is the closest point on Γ to x, (4.9) yields

$$|\delta(x)| \le \int_x^x \|\nabla\delta(t)dt\| \le \epsilon|x-x \;| \le \epsilon|x-y_0|.$$

From the two latter estimates it follows that

$$\left|\int_{\partial\Omega_\epsilon} \delta(x)\frac{\partial k_n(x-y_0)}{\partial n_x}dS_x\right| \le \epsilon \text{ const} \int_{\partial\Omega_\epsilon} \frac{1}{|x-y_0|^{n-2}}dS_x \le \epsilon \text{ const}\|\partial\Omega_\epsilon\|. \quad (4.12)$$

Taking a limit in (4.10) as $\epsilon \to 0$ and using (4.11), (4.12), and (4.9(c)), we obtain (4.3). This completes the proof of our Theorem, modulo the Assertion (4.7).

To prove the Assertion, assume at first that $\hat{T} = k_n(x-x_0)$ for some $x_0 \in \Omega$, i.e. $T = \delta_{x_0}$. Then, by excising a ball $B_\epsilon = \{x: |x-x_0| < \epsilon\}$, using Green's identity in $\Omega\backslash B_\epsilon$ and taking a limit, we find that for each $\phi \in C^\infty(\Omega)$.

$$\int_\Gamma\left\{\phi\frac{\partial\hat{T}}{\partial n} - \frac{\partial\phi}{\partial n}\hat{T}\right\}ds = \int_{\partial B_\epsilon}(\phi\frac{\partial\hat{T}}{\partial n} - \frac{\partial\phi}{\partial n}\hat{T})dS_\epsilon - \int_{\Omega\backslash B_\epsilon} \hat{T}\Delta\varphi dv \to \phi(x_0) -$$

$$-\int_{\Omega} k_n(x-x_0)\Delta\phi dx = \langle\phi,T\rangle - \langle\Delta\phi,\hat{T}\chi_{\Omega}\rangle, \tag{4.13}$$

as $\epsilon \to 0$. For an arbitrary T, using (4.13) we obtain

$$\int_{\Gamma}\left\{\phi\frac{\partial\hat{T}}{\partial n} - \frac{\partial\phi}{\partial n}\hat{T}\right\}dS_x = \left\langle \int_{\Gamma}\left\{\phi(x)\frac{\partial k_n(x-y)}{\partial n_x} - \frac{\partial\phi(x)}{\partial n_x}k_n(x-y)\right\}dS_x,\ T(y)\right\rangle$$

$$= \langle\phi(y) - \int_{\Omega} k_n(x-y)\Delta\phi\ dx, T(y)\rangle = \langle\phi,T\rangle - \langle\Delta\phi,\hat{T}\chi_{\Omega}\rangle$$

and the proof of the assertion is complete.

Corollary 4.3. Let Ω *be a quadrature domain with an analytic boundary* Γ. *Then, the following hold:*

(i) The Schwarz potential $U_\Gamma(x)$ *of* Γ *is real-analytic in* $\Omega\setminus\text{supp } T$.

(ii) The gravitational potential $\hat{\Omega}(x)$ *of* Ω *can be analytically continued inside* Ω *and is real-analytic in* $\Omega\setminus\text{supp } T$. *Moreover,*

$$\hat{\Omega}(x) = \frac{1}{2n}|x|^2 - \frac{1}{n}U_\Gamma(x) + \hat{T}(x),\ x \in \Omega. \tag{4.14}$$

Conversely, assume that there is a distribution T: $\text{supp } T \subset \Omega$ *such that* $\frac{1}{n}U_\Gamma(x) - \hat{T}(x)$ *is harmonic in* Ω *(i.e.* $(\Delta U_\Gamma = nT$ *in* Ω*, in the distributional sense). Then,* Ω *is a quadrature domain with respect to the distribution* T.

Proof. If Ω is a quadrature domain, then (4.4) holds and therefore according to the classical result on harmonic continuation ([Ke, Ch. X, §5, Thm. VI]) we conclude that $\hat{T}$ defines a harmonic continuation of $\hat{\Omega}$ into Ω. Also, according to (4.3) and the definition of the Schwarz potential we obtain

$$u(x) \overset{\text{def}}{=} \frac{1}{n}U_{\Gamma}(x) - \hat{T}(x) + u_0(x) \equiv 0 \text{ on } \Gamma. \tag{4.15}$$

Since u is harmonic in Ω, from the uniqueness of the solution of the Cauchy problem, it follows that $u = 0$. Hence $\frac{1}{n}U_{\Gamma} = \hat{T} - u_0$ in Ω. Finally, (4.14) follows from (4.15) and (4.5). The converse statement is an obvious consequence of Theorem 4.2.

<u>Remark 4.4</u>. If $n = 2$, then taking the gradients in (4.3) and conjugating we obtain (see §1) that

$$\bar{z} = 2(\overline{\nabla\hat{T}} - \overline{\nabla u_0}) \overset{\text{def}}{=} S(z) \text{ on } \Gamma,$$

where $S(z)$ is analytic in $\Omega\backslash\text{supp } T$. From this, using conformal mapping and the symmetry principle one can readily derive that $S(z)$ can be analytically continued across Γ and, therefore, locally Γ is an analytic arc (see [AS], [Sa 1]). Unfortunately, we have not been able to obtain such a remarkable regularity for $n \geq 3$. Note that the problem (4.3) is a typical "free boundary problem", i.e. a problem of finding Ω compatible with the overdetermining boundary data (4.3). So, an open question concerning the "smoothness" of the solution of such a free boundary problem for $n \geq 3$ is of major importance. On the other hand, if one assumes that $\Gamma = \partial\Omega$ is already of class C^1, then the analyticity of Γ readily follows from much more general results of Kinderlehrer and Nirenberg [KN], [Fr]. Thus, the problem is to obtain at least some apriori regularity of the boundary of a quadrature domain. For $n = 2$ this problem was recently solved completely by M. Sakai [Sa 4] in a remarkable paper.

To illustrate Theorem 4.2 and Corollary 4.3 let us now discuss a quadrature identity for ellipsoids in greater detail. For the sake of clarity we restrict ourselves to the case $n = 3$. To avoid tedious calculations we will use the following consequence of a classical result of MacLaurin (see [Ma, p. 60]). Recall that a family of ellipsoids $\mathcal{E}_\lambda$:

$$\mathcal{E}_\lambda: \frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \frac{x_3^2}{a_3^2+\lambda} \leq 1, a_1 \geq a_2 \geq a_3 > 0, \ -a_3^2 \leq \lambda < +\infty$$

is called a <u>confocal family</u> for the ellipsoid

$$\mathcal{E}_0: \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq 1.$$

As $\lambda \to +\infty$, $\mathcal{E}_\lambda$ becomes nearly a ball and as $\lambda \to -a_3^2$, $\mathcal{E}_\lambda$ approaches the flat ellipse (so-called <u>focal</u> <u>ellipse</u> of the family $\mathcal{E}_\lambda$).

$$E: \quad \frac{x_1^2}{a_1^2-a_3^2} + \frac{x_2^2}{a_2^2-a_3^2} \le 1, x_3 = 0. \tag{4.16}$$

Incidentally, for $n=2$, $\{\mathcal{E}_\lambda\}$ is a family of ellipses with fixed foci (see [Ke, Ch. VII, §4]).

Lemma 4.5. For any $u \in L_h^1(\mathcal{E}_0)$, *the mean value*

$$\frac{1}{\mathrm{Vol}(\mathcal{E}_\lambda)} \int_{\mathcal{E}_\lambda} u dx$$

is independent of λ.

Remark. An elegant proof of this result and of a more general mean value theorem due to L. Asgeirsson and dealing with solutions of partial differential equations of a special type can be found in [CH, Ch. VI, §16]. Here, we give an independent proof.

Proof. We start with a number of reductions.

(i) Without loss of generality we can assume that u is a homogeneous harmonic polynomial, since harmonic polynomials are dense in $L_h^1(\mathcal{E}_0)$. Moreover, it actually suffices to take $u = (x\circ\mu)^m$, $m = 0,1,\dots,$ where $\mu=(\mu_1,\mu_2,\mu_3) \in V = \{\mu:\sum_1^3 \mu_k^2=0\} \subset \mathbb{C}^3$. Here, $x\circ\mu = \sum_1^3 x_k\mu_k, x = (x_1,x_2,x_3) \in \mathbb{R}^3$, $\mu = (\mu_1,\mu_2,\mu_3) \in \mathbb{C}^3$. The latter fact is well-known but for the reader's convenience we shall indicate the argument.

Introduce into each of the vector spaces H_m (homogeneous polynomials of degree m with complex coefficients) the "Fischer" inner product ([Fi], [SW, Ch. IV])

$$<f,g> = \sum_{|\alpha|=m} \alpha! f_\alpha \overline{g_\alpha} \ , f,g \in H_m$$

$$f = \sum_{|\alpha|=m} f_\alpha x^\alpha, \ g = \sum_{|\alpha|=m} g_\alpha x^\alpha \ , f_\alpha, g_\alpha \in \mathbb{C}.$$

Here, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$ is the standard multi-index notation (cf. [Hö]).

Then for any $\mu \in \mathbb{C}^3$, a simple calculation yields

$$<f, \frac{(x\circ\bar{\mu})^m}{m!}> = f(\mu),$$

for all $f \in H_m$. Now, if $u \in H_m$ is harmonic and $<u,(x\circ\mu)^m> = 0$ for all $\mu \in V$, then u vanishes on the variety V. Hence, by Hilbert's Nullstellensatz, $u(\mu) = (\sum_1^3 \mu_k^2)q(\mu), q \in H_{m-2}$. According to the theorem of E. Fischer - and this is the main point of Fischer's formalism, - the operator "multiplication by $(\sum_1^3 \mu_k)^2$" from H_m to H_{m+2} and $\Delta: H_{m+2} \to H_m$ are mutually adjoint with respect to the inner product < , >. So,

$$\langle\Delta u,q\rangle = 0 = \langle\Delta\{(\sum_1^3 \mu_k^2)q(\mu)\},q(\mu)\rangle = \langle u,u\rangle = \|u\|^2,$$

i.e. $u = 0$. Thus, $\{(x\circ\mu)^m,\ \mu \in V\}$ span $\ker \Delta$ in H_m.

(ii) Let $\mathcal{E}$ be an ellipsoid confocal with $\mathcal{E}_0$. Let b_1, b_2, b_3: $a_i^2 - b_i^2 = \lambda$, $i = 1,2,3$ be its semiaxes. Then, a simple change of variables yields

$$\frac{1}{\mathrm{Vol}(\mathcal{E}_\lambda)} \int_{\mathcal{E}_\lambda} u(x_1,x_2,x_3)dx = \frac{1}{\mathrm{Vol}(B)} \int_B u(b_1x_1,b_2x_2,b_3x_3)dx,$$

where $B = \{x: |x| < 1\}$ denotes the unit ball in $\mathbb{R}^3$. Thus, taking into account the reduction (i) it suffices to show that for all m

$$\int_B (a_1\mu_1x_1 + a_2\mu_2x_2 + a_3\mu_3x_3)^m dx = \int_B (b_1\mu_1x_1 + b_2\mu_2x_2 + b_3\mu_3x_3)^m dx$$

for all $\mu \in V$, whenever $a_i^2 - b_i^2 = \lambda$ for all $i = 1,2,3$. Set $\zeta_i = a_i\mu_i, \omega_i = b_i\mu_i, i = 1,2,3$. Then, $\sum_1^3(\zeta_i^2 - \omega_i^2) = \lambda\sum_1^3 \mu_i^2 = 0$ for all $\mu \in V$. So, the lemma is reduced to the following purely algebraic assertion.

<u>Assertion</u>. *The polynomial* $P(t_1,t_2,t_3) \in H_m$ *defined by*

$$P(t) = \int_B (t\circ x)^m dx$$

depends only on $(\sum_1^3 t_k^2)$, $t = (t_1, t_2, t_3) \in \mathbb{C}^3$.

To verify the assertion we observe that since $P(t)$ is obviously invariant under the group of rotations in $\mathbb{R}^3$, $P(t) = c = \text{const}$ on the unit sphere S^2 in $\mathbb{R}^3$. Hence, $P(t) - c$ is divisible by $(\sum_1^3 t_k^2) - 1$, i.e. $P(t) - c = (\sum_1^3 t_k^2 - 1)q(t)$. Noting that $c = q(0)$, we conclude that $P(t)$ itself is divisible by $(\sum_1^3 t_k^2)$ in $\mathbb{C}[t_1, t_2, t_3]$. Thus $P(t) = (\sum_1^3 t_k^2)^r Q(t)$, where $Q(t)$ is relatively prime to $(\sum_1^3 t_k^2)$. Since $Q = \text{const}$ on S^2, repeating the above argument we obtain that Q must be a constant. The proof of the Lemma is now complete.

Theorem 4.6. Let $\mathcal{E}_0$ *be the ellipsoid*

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \le 1, \; a_1 > a_2 > a_3.$$

Then, for any $u \in L_h^1(\mathcal{E}_0)$ *the following quadrature identity holds*

$$\int_{\mathcal{E}_0} u\,dx = \frac{2a_1a_2a_3}{\left((a_1^2-a_3^2)(a_2^2-a_3^2)\right)^{\frac{1}{2}}} \int_E u(x_1,x_2,0) \left(1 - \frac{x_1^2}{a_1^2-a_3^2} - \frac{x_2^2}{a_2^2-a_3^2}\right)^{\frac{1}{2}} dx_1 dx_2. \tag{4.17}$$

(Here, E is defined by (4.16)).

Proof. Without loss of generality we can assume u to be harmonic in $\overline{\mathcal{E}_0}$. According to Lemma 4.5 we have for all

λ: $-a_3^2<\lambda\leq 0$

$$\frac{1}{\mathrm{Vol}\ (\mathcal{E}_0)}\int_{\mathcal{E}_0} u dx = [(4\pi/3)(a_1a_2a_3)]^{-1}\int_{\mathcal{E}_0} u dx = \frac{1}{\mathrm{Vol}\ (\mathcal{E}_\lambda)}\int_{\mathcal{E}_\lambda} u dx. \tag{4.18}$$

Since

$$\int_{\mathcal{E}_\lambda} u(x_1,x_2,x_3)dx = \int_{\mathcal{E}_\lambda}[u(x_1,x_2,x_3) - u(x_1,x_2,0)]dx + \int_{\mathcal{E}_\lambda} u(x_1,x_2,0)dx,$$

$$\max|u(x_1,x_2,x_3) - u(x_1,x_2,0)| \leq \max\|\nabla u\|(a_3^2+\lambda)^{\frac{1}{2}} \to 0, \text{ as } \lambda\downarrow -a_3^2,$$

we obtain from (4.18), that

$$\int_{\mathcal{E}_0} u dx = (4/3)\pi a_1a_2a_3 \lim_{\lambda\downarrow -a_3^2}\left\{\frac{1}{\mathrm{Vol}(\mathcal{E}_\lambda)}\int_{\mathcal{E}_\lambda} u(x_1,x_2,0)dx\right\}. \tag{4.19}$$

For fixed x_1,x_2 set $A_\lambda = [(1-x_1^2/(a_1^2+\lambda) - x_2^2/(a_2^2+\lambda))(a_3^2+\lambda)]^{\frac{1}{2}}$.

Applying Fubini's Theorem, we compute

$$\int_{\mathcal{E}_\lambda} u(x_1,x_2,0)dx = \int_{\mathcal{E}_\lambda\cap\{x_3=0\}} u(x_1,x_2,0)\left\{\int_{-A_\lambda}^{A_\lambda} dx_3\right\} dx_1dx_2$$

$$= 2(a_3^2+\lambda)^{\frac{1}{2}}\left\{\int_E u(x_1,x_2,0)\left(1 - x_1^2/(a_1^2+\lambda) - x_2^2/(a_2^2+\lambda)\right)^{\frac{1}{2}}dx_1dx_2 +\right.$$

$$+ \int_{\mathcal{E}_\lambda \cap (\{x_3=0\}\setminus E)} u(x_1,x_2,0)\left(1 - x_1^2/(a_1^2+\lambda) - x_2^2/(a_2^2+\lambda)\right)^{\frac{1}{2}} dx_1 dx_2 \Big\}. \tag{4.20}$$

As $\mathrm{Vol}(\mathcal{E}_\lambda) = (4\pi/3)[(a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda)]^{\frac{1}{2}}$ and Area $(\mathcal{E}_\lambda \cap \{x_3=0\}\setminus E) \to 0$ as $\lambda \to -a_3^2$, we obtain from (4.19) and (4.20) the desired formula (4.17).

Remark 4.7. It is worth mentioning that if $\mathcal{E}_0$ is an ellipsoid of rotation, i.e. $a_1 > a_2 = a_3$, the quadrature formula (4.17) transforms into the following

$$\int_{\mathcal{E}_0} u dx = (\pi/\epsilon) a_1 a_2^2 \int_{-\epsilon}^{\epsilon} u(x_1,0,0)(1-x_1^2/\epsilon^2) dx_1$$

$$= (\pi/\epsilon^3) a_1 a_2^2 \int_{-\epsilon}^{\epsilon} u(x_1,0,0)(\epsilon^2-x_1^2) dx_1, \text{ where } \epsilon = \sqrt{a_1^2-a_2^2}. \quad 4.21)$$

Remark 4.8. For $n = 2$, the analog of (4.17) i.e. the quadrature formula for ellipses is well-known (e.g. see [Da 2, p. 133]).

We recall that if $\mathcal{E}_0$ is the ellipsoid, then inside $\mathcal{E}_0$ its gravitational potential $\hat{\mathcal{E}}_0$ is equal to a quadratic polynomial $Q(x) = D + Ax_1^2 + Bx_2^2 + Cx_3^2$, $D < 0$, $A,B,C > 0$, $A + B + C = \frac{1}{2}$, (see e.g. [Ke, p. 194]). An elegant derivation of this fact based on the theorem of Newton, which states that the gravitational potential of an elliptic shell vanishes inside the shell, can be found in [DF].

Corollary 4.9. *Let* $\mathcal{E}_0$ *be the ellipsoid* $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \le 1$, $a_1 > a_2 > a_3$ *and let*

$$T = 2a_1a_2a_3(a_1^2-a_2^2)^{-\frac{1}{2}}(a_2^2-a_3^2)^{-\frac{1}{2}}[1-x_1^2/(a_1^2-a_2^2)-x_2^2/(a_2^2-a_3^2)]^{\frac{1}{2}}\chi_E dx_1 dx_2$$

be its quadrature distribution (4.17). (E is defined by (4.16)).

Then the following hold.

(i) The Schwarz potential $U_{\partial\mathcal{E}_0}$ *is harmonically continuable along all paths in* $\mathbb{R}^3\backslash\partial E$ *and*

$$\tfrac{1}{3}U_{\partial\mathcal{E}_0}(x) = \hat{T}(x) + h(x), \tag{4.22}$$

where $h(x_1,x_2,x_3) = \frac{1}{6}|x|^2 - Q(x)$ *is a harmonic polynomial of degree* 2.

(ii) The gravitational potential $\hat{\mathcal{E}}_0(x_1,x_2,x_3)$ *can be harmonically continued into* $\mathcal{E}_0\backslash\partial E$. *Moreover,*

$$\hat{\mathcal{E}}_0 \equiv \hat{T} \text{ on } \partial\mathcal{E}_0 \text{ and}$$

$$\hat{\mathcal{E}}_0 = \hat{T} \text{ in } \mathbb{R}^3\backslash\mathcal{E}_0. \tag{4.23}$$

(iii) Let $\omega(x)$ *be the equilibrium potential of the ellipsoidal conductor of charge* μ *distributed over* $\partial\mathcal{E}_0$ *and such that*

$$\omega = 1 \quad \text{in } \mathcal{E}_0;$$

$$\Delta\omega = 0 \quad \text{in } \mathbb{R}^3\backslash\partial\mathcal{E}_0,$$

$$\omega \in C(\mathbb{R}^3),$$

$$|x|\omega \to \mu \text{ as } |x| \to \infty, \tag{4.24}$$

(cf. [Ke 1], Ch. VII, §5). Then everywhere in $\mathbb{R}^3\backslash\overline{\mathcal{E}_0}$

$$\omega(x) = \frac{1}{2D}\left(2\hat{T} - (x\circ\nabla\hat{T})\right). \tag{4.25}$$

<u>*Proof*</u>. From (4.14), (4.17) and the definition of $Q(x)$ we obtain (4.22). Also, (4.17) yields (4.23). To establish (4.25), observe that the function

$$v(x) \overset{\text{def}}{=} x\circ\nabla\hat{T} = x_1\frac{\partial\hat{T}}{\partial x_1} + x_2\frac{\partial\hat{T}}{\partial x_2} + x_3\frac{\partial\hat{T}}{\partial x_3}$$

is real-analytic outside of supp T. Moreover, a direct calculation reveals that

$$\Delta v = 2\Delta\hat{T} + x_1\frac{\partial\Delta\hat{T}}{\partial x_1} + x_2\frac{\partial\Delta\hat{T}}{\partial x_2} + x_3\frac{\partial\Delta\hat{T}}{\partial x_3} = 0$$

everywhere in $\{\mathbb{R}^3\backslash\text{supp } T\}$. Also, we have on $\partial\mathcal{E}_0$

$$v(x) = x_1\frac{\partial\hat{\mathcal{E}}_0}{\partial x_1} + x_2\frac{\partial\hat{\mathcal{E}}_0}{\partial x_2} + x_3\frac{\partial\hat{\mathcal{E}}_0}{\partial x_3} = 2(Ax_1^2+Bx_2^2+Cx_3^2) =$$

$$2(Q - D) = 2(\hat{\mathcal{E}}_0 - D) = 2(\hat{T} - D).$$

Hence, $2\hat{T} - v$ is harmonic in $\mathbb{R}^3\backslash\overline{\mathcal{E}_0}$, vanishes at infinity, and equals $2D$ on $\partial\mathcal{E}_0$. So, the function $\frac{1}{2D}(2\hat{T} - v)$ satisfies (4.24) and, therefore (4.24) holds in $\mathbb{R}^3\backslash\mathcal{E}_0$.

Remark 4.10. Results of similar nature to Corollary 4.9 for families of confocal quadrics were obtained using different methods by the school of V.I. Arnold, see [Ar], [Gi], [VS].

Let us now discuss the quadrature identity for cylinders (observe that the measure in this q.i. is not compactly supported in Ω).

Theorem 4.11. Let $\Omega = \{x : x_1^2 + x_2^2 \le 1\}$ *be a circular cylinder. Then, for any* $u \in L_h^1(\Omega)$ *integrable on the* x_3*-axis the following quadrature identity holds*

$$\int_{\Omega} u dx = \pi \int_{-\infty}^{\infty} u(0,0,x_3)\, dx_3 \tag{4.26}$$

Proof. Let $\mathcal{E}_\lambda = \{x : x_1^2 + x_2^2 + (x_3^2/\lambda^2) \leq 1\}$ be a family of ellipsoids of rotation inscribed into Ω. Then, from (4.21) it follows, that

$$\int_{\Omega} u dx = \lim_{\lambda \uparrow \infty} \int_{\mathcal{E}_\lambda} u dx =$$

$$= \lim_{\lambda \uparrow \infty} \pi\lambda(\lambda^2-1)^{-3/2} \int_{-(\lambda^2-1)^{\frac{1}{2}}}^{(\lambda^2-1)^{\frac{1}{2}}} u(0,0,x_3)\ (\lambda^2-1-x_3^2) dx_3 \tag{4.27}$$

Since $\chi_{[-(\lambda^2-1)^{\frac{1}{2}},(\lambda^2-1)^{\frac{1}{2}}]}\lambda(\lambda^2-1)^{-3/2}\ (\lambda^2-1-x_3^2) \to 1$ for all x_3 as $\lambda \uparrow \infty$, applying Lebesgue's dominated convergence theorem to (4.27), we obtain (4.26).

We note that with a little bit more work one can prove (4.26) directly, without making use of the ellipsoid quadrature.

Remark 4.12. Karp (unpublished) has shown that formula (4.26) can be extended in an appropriate form to general cylinders. For the elliptic cylinders $\Omega : \{x : x_1^2/a_1^2 + x_2^2/a_2^2 \leq 1,\ a_1 > a_2\}$, one can adapt the above argument, using (4.17) instead of (4.21) and obtain for $u \in L_h^1(\Omega)$, the following identity

$$\int_{\Omega} udx = \frac{2a_1a_2}{a_1^2-a_2^2} \int_{-\infty}^{\infty} \int_{-(a_1^2-a_2^2)^{\frac{1}{2}}}^{(a_1^2-a_2^2)^{\frac{1}{2}}} u(x_1,0,x_3)\ (a_1^2-a_2^2-x_1^2)^{\frac{1}{2}} dx_1 dx_3. \quad (4.28)$$

As is seen from (4.14), (4.22) and (4.28) Schwarz potentials of ellipsoids and elliptic cylinders are analytic in the complements of those solids. This is exactly the reason why those regions turn out to be "null quadrature domains" (cf. [Sa 2], [FS], [Sh 1], [Sha 1,2,3]. More precisely, the following result holds.

<u>*Theorem 4.13*</u>. *Let* Ω *be either an ellipsoid or an elliptic cylinder. Then,* $\Omega^1 = \mathbb{R}^3\setminus\overline{\Omega}$ *is a "null quadrature domain" for* L_h^1, i.e.

$$\int_{\Omega^1} udx = 0, \quad \textit{for all} \quad u \in L_h^1(\Omega^1).$$

<u>*Proof*</u>. For the sake of brevity we shall conduct the argument for ellipsoids. It was first proved in [FS], but the argument we present here is based on a different idea and simpler than the one in [FS]. Let us start with a simple assertion.

<u>Assertion</u>. *Let* $\Omega^1=\{x : x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 > 1\}$. *If* $u \in L_h^1(\Omega^1)$, *then* $u = O(|x|^{-4})$ *at* ∞.

Indeed, applying Kelvin's transformation $x \mapsto y=T(x)=x/|x|^2$, which maps Ω^1 onto a bounded domain D, and transforms the volume element dx into $|y|^{-6}dy$, we obtain that $u \in L^1(\Omega^1)$

transforms into $u(T^{-1}y)|y|^{-6} \in L^1(D)$. As u is regular at ∞ and $T(\infty) \in D$, the assertion follows.

Denote by U the Schwarz potential of $\partial\Omega$ and let $B_R = \{x : |x| < R\}$. Then, using the assertion, (4.22) and Green's formula, we find

$$6\int_{\Omega^1} udx = 6 \lim_{R\uparrow\infty} \int_{B_R\setminus\Omega} udx = \lim_{R\uparrow\infty} \int_{B_R\setminus\Omega} u\Delta(|x|^2)dx =$$

$$= \lim_{R\uparrow\infty} \left\{ \int_{\partial B_R} \left[u \frac{\partial|x|^2}{\partial n} - |x|^2 \frac{\partial u}{\partial n} \right] dS \right\} - \int_{\partial\Omega} \left[u\frac{\partial|x|^2}{\partial n} - |x|^2 \frac{\partial u}{\partial n} \right] dS =$$

$$= \lim_{R\uparrow\infty} \left\{ \int_{\partial B_R} \left[u\frac{\partial U}{\partial n} - U \frac{\partial u}{\partial n} \right] dS - \int_{\partial\Omega} \left[u \frac{\partial U}{\partial n} - U \frac{\partial u}{\partial n} \right] dS \right\}$$

$$+ \lim_{R\uparrow\infty} \int_{\partial B_R} \left[u \frac{\partial(|x|^2-U)}{\partial n} - (|x|^2 - U) \frac{\partial u}{\partial n} \right] dS = 0 +$$

$$+ \lim_{R\uparrow\infty} \int_{\partial B_R} \left[u \frac{\partial(|x|^2-U)}{\partial n} - (|x|^2 - U) \frac{\partial u}{\partial n} \right] dS = 0.$$

Remark 4.14. It is much less obvious [FS], that ellipsoids are the only compact solids whose complement is a "null quadrature domain." Combining this fact with the proof of Theorem 4.13 given above, we conclude that ellipsoidal surfaces are the only simple closed surfaces whose Schwarz potentials are analytic at all points of their unbounded complementary component with the possible exception of ∞, and are $O(|x|^2)$ as $x \to \infty$.

§5. Further developments

I. As follows from Corollary 4.9 the singularity set of the Schwarz potential U_Γ of the ellipsoidal surface

$\Gamma: \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$ lies on the 1-dimensional ellipse which bounds the focal ellipse $E: \frac{x_1^2}{a_1^2 - a_3^2} + \frac{x_2^2}{a_2^2 - a_3^2} \le 1$. In reality, the density of the distribution T on the elliptic lamina E is analytic in the interior of E. Therefore, the potential $\hat{T}$ can be analytically continued across the lamina. However, this continuation ceases to be the potential of the lamina because that potential must have a break in its normal derivative on the lamina. Thus, the function cannot be single-valued and has non-zero periods, as we continue it along a path surrounding the ellipse which bounds the lamina E. So, $\hat{T}$ and therefore U_Γ can be analytically continued everywhere inside Γ except for the boundary of the focal ellipse ∂E. Note that the situation is very much like that for the plane ellipses (see §3), where the Schwarz function is analytically continuable (but not single-valued!) into the whole plane excluding two foci. Therefore, the Conjecture 3.4 from §3 applied to the specific situation of the ellipsoid transforms into the following.

Can the solution of the Cauchy problem (3.3) on Γ *with polynomial data be analytically continued everywhere in* $\mathbb{R}^3$ *except for points of the focal 1-dimensional ellipse* ∂E?

H. Shahgholian [Sha 1] has recently answered it in the affirmative for arbitrary n.

Here is the sketch of his argument.

Let $\mathcal{E}_a = \{x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2} + \dots + \frac{x_n^2}{a_n^2} < 1,\ a_1 \geq a_2 \geq \dots \geq a_n > 0\}$

be an n-dimensional ellipsoid and let

$$E = \{x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2 - a_n^2} + \dots + \frac{x_{n-1}^2}{a_{n-1}^2 - a_n^2} < 1,\ x_n = 0\}$$

be the "focal ellipsoid" of $\mathcal{E}_a$.

For P a polynomial, define

$$H(y) := H^P(y) := \int_{\mathcal{E}_a} \frac{P(x)}{|x-y|^{n-2}}\, dx.$$

$H(y)$ is of class $C^1(\mathbb{R}^n)$ and harmonic in $\mathbb{R}^n \setminus \overline{\mathcal{E}}_a$. Extending classical results of Newton and Dirichlet ($P = 1$, $n = 3$) and Ferrers [Fer] (arbitrary P, $n = 3$) Shahgholian has shown (cf. [Gi]) that inside E_a, $H(y)$ coincides with a polynomial Q of degree $\deg P + 2$ (clearly, $\Delta Q = \text{const} \cdot P$ inside $\mathcal{E}_a$) and, moreover, $H(y)$ is <u>harmonically</u> continuable across $\Gamma = \partial \mathcal{E}_a$ inside $\mathcal{E}_a$ to $\mathcal{E}_a \setminus E$. A finer argument shows that $H(y)$ is continuable along all paths that do not meet ∂E.

Now, consider the Cauchy problem.

$$\begin{cases} \Delta v = 0 & \text{near } \partial \mathcal{E}_a \\ v \equiv P & \text{on } \partial \mathcal{E}_a \text{ (P is a polynomial).} \end{cases} \tag{5.1}$$

Let u be the potential of the measure $c\,(\Delta P)\,\chi_{\mathcal{E}_a}\,dx$, where the constant c is so chosen that

$$\Delta u = \Delta P \quad \text{in } \mathcal{E}_a$$

($\chi_{\mathcal{E}_a}$ is the characteristic function of $\mathcal{E}_a$).

Then,

$$u = \begin{cases} P + H_1 & \text{in } \mathcal{E}_a \\ H & \text{in } \mathbb{R}^n \setminus \mathcal{E}_a \end{cases}$$

where H_1, $H := H^{\Delta P}$ are harmonic in $\mathcal{E}_a$ and $\mathbb{R}^n \setminus \mathcal{E}_a$ respectively. In view of (5.1) we have

$$H_1 + v \equiv H \text{ on } \partial\,\mathcal{E}_a$$

i.e. $H_1 + v$ and H have matching Cauchy data of the first order on $\partial\,\mathcal{E}_a$. Hence, $v \equiv H - H_1$ on $\partial\,\mathcal{E}_a$ and so is harmonically continuable inside E_a as far as H is, i.e. to $\mathcal{E}_a \setminus \partial\,E$.

H_1 is harmonic in $\mathcal{E}_a$ and since u is equal to a polynomial inside $\mathcal{E}_a$ we obtain from (5.1) that H_1 has polynomial Dirichlet data on $\partial\,\mathcal{E}_a$. An argument similar to that in Lemma 3.8 shows that H_1 is in fact a polynomial and therefore extends harmonically to the whole in $\mathbb{R}^n$. Thus, $v = H - H_1$ extends to $\mathbb{R}^n \setminus \partial E$.

Most recently, using different methods, Johnsson [Jo 2] has obtained the following remarkable generalization of Shahgholian's result.

Let $\hat{\Gamma}_a$ denote the "complexified" ellipsoidal surface (where $\Gamma_a = \partial \mathcal{E}_a \subset \mathbb{R}^n$) :

$$\hat{\Gamma}_a : = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{z_j^2}{a_j^2} = 1\}$$

and consider the "complexified" family of confocal ellipsoidal surfaces in $\mathbb{C}^n$

$$\hat{\Gamma}_a^{\lambda} : = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{z_j^2}{a_j^2+\lambda} = 1, \ \lambda \in \mathbb{C}\}.$$

Observe that (for fixed a) the $\hat{\Gamma}_a^{\lambda}$ for different λ intersect in $\mathbb{C}^n$. Then, the envelope Σ of all those surfaces, i.e. a surface tangent to all surfaces $\hat{\Gamma}_{\lambda}$ in the family, is a certain irreducible (if $n \geq 3$!) algebraic variety which intersects $\mathbb{R}^n$ in the focal ellipsoid of Γ_a. (In fact, Σ is a ruled surface which is everywhere characteristic with respect to the Laplace operator and can be calculated explicitly.)

Then, *the solution of every Cauchy problem*

$$\begin{cases} \Delta v = 0 \quad \text{near} \quad \partial \mathcal{E}_a = \Gamma_a = \hat{\Gamma}_a \cap \mathbb{R}^n, \\ v \equiv f \quad \text{on} \quad \Gamma_a, \end{cases}$$

where f *extends to* $\mathbb{C}^n$ *as an entire function is analytically extendible along every path in* $\mathbb{C}^n$ *which does not meet* Σ.

Note that when $a_1 = \ldots = a_n$, i.e. Γ_a is a sphere, Σ

becomes an isotropic cone $\{z : \sum_1^n z_j^2 = 0\}$ and we obtain an earlier theorem of G. Johnsson [Jo 2] mentioned in §3.

II. As we noted before, unfortunately Schwarz potentials in $\mathbb{R}^n$ do not satisfy all those nice geometric properties which make the Schwarz function such a useful instrument in $\mathbb{R}^2$. In particular, in $\mathbb{R}^2$, $S(z)$ yields the Schwarzian symmetry with respect to the analytic arc that it defines. But in $\mathbb{R}^n$, $\text{grad } U_\Gamma: \mathbb{R}^n \to \mathbb{R}^n$ does not satisfy this property even when Γ is a plane (see §2,I). (This is hardly surprising, since there there is no analogous reflection principle for general analytic hypersurfaces in more than two dimensions [KS 2].) Moreover, in $\mathbb{R}^2$ the mapping $R : z \mapsto \text{grad } U_\gamma(z) = \overline{S(z)}$ is an involution, i.e. $R \circ R \equiv$ identity at all points z where this composition is defined, in particular, this implies that at all such points R is one-to-one. Therefore, one can raise the following question.

Let $\Gamma \subset \mathbb{R}^n$ *be a real-analytic surface and let* U_Γ *be its Schwarz potential. Define the mapping* $R : x \mapsto$ *grad* $U_\Gamma(x) :$ $= \left(\frac{\partial U_\Gamma}{\partial x_1}, \ldots, \frac{\partial U_\Gamma}{\partial x_n}\right)$ *at all points* $x \in \mathbb{R}^n$ *where* U_Γ *is harmonic. Suppose* $\Omega \subset \mathbb{R}^n$, $\Omega \supset \Gamma$ *is invariant under the mapping* R, i.e. $R(\Omega) \subset \Omega$. *Is* R *one-to-one on* Ω?

If $\Gamma \subset \mathbb{R}^3$ is a plane, e.g. $\Gamma := \{x : x_3 = 0\}$, then $U_\Gamma = \frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)$ and so $R(x) = (x_1, x_2, -2x_3)$ is a composition of a symmetry about Γ followed by dilatation with the coefficient 2 in the direction of the x_3-axis normal to Γ.

It turns out that for arbitrary surfaces $\Gamma \subset \mathbb{R}^n$, R behaves infinitesimally in the same manner sufficiently close to Γ. More precisely, near $x^\circ \in \Gamma$ for all x lying on the normal n_x towards Γ at x°, $R(x) := A\,x + O(|x - x^\circ|)$, where the linear transformation A is a composition of symmetry with respect to the tangent plane to Γ at x° and a dilatation with the coefficient $(n - 1)$ in the n_{x° direction. Indeed, assuming for the sake of simplicity of notation $n = 3$, we have

$$A = \text{Hessian } U_\Gamma\,(x^\circ) := \left(\frac{\partial^2 U_\Gamma}{\partial x_i \partial x_j}\bigg|_{x^\circ}\right)_{i,j=1,2,3}$$

Since U_Γ is harmonic, $\operatorname{tr} A = 0$. Without loss of generality let $x^\circ = 0$.

Let τ be the tangent plane to Γ at x°. Since $x^i - \frac{\partial U_\Gamma}{\partial x_i} = 0$, for $i = 1,2,3$ on Γ, the equation of τ can be written in one of the following forms (all derivatives are evaluated at the origin):

$$\left(1 - \frac{\partial^2 U}{\partial x_1^2}\right) x_1 - \frac{\partial^2 U_\Gamma}{\partial x_1 \partial x_2} x_2 - \frac{\partial^2 U_\Gamma}{\partial x_1 \partial x_3} x_3 = 0$$

$$- \frac{\partial^2 U_\Gamma}{\partial x_1 \partial x_2} x_1 + \left(1 - \frac{\partial^2 U_\Gamma}{\partial x_2^2}\right) x_2 - \frac{\partial^2 U_\Gamma}{\partial x_2 \partial x_3} x_3 = 0 \qquad (5.2)$$

$$- \frac{\partial^2 U_\Gamma}{\partial x_1 \partial x_3} x_1 - \frac{\partial^2 U_\Gamma}{\partial x_1 \partial x_3} x_2 + \left(1 - \frac{\partial^2 U}{\partial x_3^2}\right) x_3 = 0.$$

Hence, all rows of the matrix $(\mathrm{Id} - A)$ are normal vectors to Γ at 0 and so rank $(\mathrm{Id}-A) = 0$, or 1. Since $\mathrm{tr}\,(\mathrm{Id}-A) = 3$, rank $(\mathrm{Id} - A) = 1$. Therefore, 1 is an eigenvalue of A of multiplicity 2. Moreover, performing a rotation if necessary we can assume that A has the diagonal form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

i.e. A is a symmetry about a certain plane τ' followed by a dilatation in the normal direction to τ' with coefficient 2. In particular, this implies that $A\,|_{\tau'} =$ identity. Observing that in view of (5.2) for $x \in \tau$ we have coordinate-wise

$$(Ax)_i = x_i \qquad i = 1, 2, 3$$

we conclude that $\tau = \tau'$. Thus, sufficiently close to Γ the answer to our question is "Yes". However, a complete answer would require much deeper study of the mapping $x \mapsto \mathrm{grad}\, U_\Gamma(x)$ (cf.[Le 3]).

Note, that for $n = 2$ (and only for $n = 2$) the dilatation reduces to the identity map and R is an involution.

The main obstacle here and that concerning regularity of the "free boundary" satisfying (4.3) seems to be our very limited knowledge about harmonic vector fields, i.e. the vector fields arising as the gradients of harmonic functions in $\mathbb{R}^n$, $n \geq 3$.

III. On the other hand, there are some very important global properties of the Schwarz function connected with the intrinsic geometry of the corresponding curve which may still hold in $\mathbb{R}^n$. More precisely, an elegant theorem of P.J. Davis ([Da 2], Ch. 14) states that in $\mathbb{R}^2$ the Schwarz function of a curve *has a meromorphic extension to* $\mathbb{C}$ *if and only if the curve is a straight line or a circle*. Plausible generalizations to $\mathbb{R}^3$ are:

(a) If the Schwarz potential of a surface Γ *is an entire harmonic function, then* Γ *is a plane.*

(b) If the Schwarz potential of a surface Γ *is a linear combination of potentials of finitely many point-masses, then* Γ *is a sphere.*

We can prove

Proposition 5.1 *Hyperplanes are the only algebraic surfaces in* $\mathbb{R}^n$ *whose Schwarz potentials are polynomials.*

Here is a sketch of the argument.

Assume that $U_\Gamma \equiv \frac{1}{2}|x|^2$ on Γ.Γ is an algebraic surface and hence there exists an irreducible polynomial $\varphi(x)$ with real coefficients such that $\Gamma \subset \{x \in \mathbb{R}^n : \varphi(x) = 0\}$. Moreover, since Γ is non-singular and

$$U_\Gamma - \frac{1}{2}|x|^2 \equiv 0 \quad \text{on } \Gamma \tag{5.3}$$

applying Hilbert's Nullstellensatz twice we obtain from (5.3) that

$$U_\Gamma(x) - \frac{1}{2}|x|^2 = r(x)\,\varphi^2(x) \tag{5.4}$$

where $r(x)$ is another polynomial. Let $\varphi = \sum_{m=0}^{\ell} \varphi_m(x)$, $\ell = \deg \varphi$, φ_m are homogeneous polynomials of degree m. We must show that $\ell = 1$. Suppose $\ell > 1$. Then, $k := \deg U_\Gamma(x) > 2$ and writing

$$U_\Gamma(x) = \sum_{m=0}^{k} h_m(x)$$

where h_m are homogeneous harmonic polynomials we obtain in view of (5.4) that $h_k(x)$ is divisible by φ_ℓ^2. This leads to an immediate contradiction because of the following fact due to M. Brelot and G. Choquet, [BC].

Lemma 5.2 *Let* P, Q *be homogeneous polynomials with real coefficients*, P *harmonic and* $Q \geq 0$. *If* P *is divisible by* Q, $P \equiv 0$.

For the reader's convenience we include the proof.

Proof of Lemma 5.2 First recall the well-known fact that for any polynomial $R : \deg R < \deg P$

$$\int_{S^{n-1}} P\,R\,d\sigma = 0$$

where $S^{n-1} = \partial B^n$ is the unit sphere and $d\sigma$ is Lebesque measure on S^{n-1}.

Hence, if $P = Q\,R$, then

$$0 = \int_{S^{n-1}} P\,R\,d\sigma = \int_{S^{n-1}} Q\,R^2\,d\sigma$$

implies that QR^2, and hence P vanish on S^{n-1}. By the maximum principle $P \equiv 0$.

Remark 5.3 Note that the above proof of Proposition 5.1 fails in $\mathbb{C}^n$ (e.g. $(\sum_1^n \alpha_j z_j)^2$, where $\sum_1^n \alpha_j^2 = 0$ is "harmonic" in $\mathbb{C}^n$).

Remark 5.4 As we have observed before (see the discussion following Conjecture 3.4) the modified Schwarz potential

$$V_\Gamma(z) := \frac{1}{2n} \sum_1^n z_j^2 - \frac{1}{n} U_\Gamma(z) \tag{5.5}$$

of a complex hypersurface $\Gamma \subset \mathbb{C}^n$ must have singularities at all characteristic points of Γ. Hence, U_Γ being entire already implies that Γ does not have characteristic points. This condition alone is not strong enough to extend Proposition 5.1 since, e.g. the "complexified sphere" $\Gamma := \{z \in \mathbb{C}^n : \sum_1^n z_j^2 = 1\}$ does not have finite characteristic points, though the Schwarz potential U_Γ is singular on the entire isotropic cone $\Sigma : \{z : \sum_1^n z_j^2 = 0\}$ (cf. formulas (2.2)). (In $\mathbb{R}^n$ we see only one point of Σ - the origin).

However, a deeper analysis shows that if we embed the sphere Γ into the projective space $\mathbb{CP}^n$, then the "projectivized" variety $\hat{\Gamma}$ does have characteristic points precisely where "projectivized" cone $\hat{\Sigma}$ hits $\hat{\Gamma}$. The Schwarz potential $U_{\hat{\Gamma}}$ (in projective coordinates) is singular on $(\hat{\Sigma} \cap \hat{\Gamma})$ and in accordance with the local theory developed by Leray [L], those singularities propagate to "finite" $\mathbb{C}^n$ along the characteristic (with respect to the Laplace operator) surface which is tangent to $\hat{\Gamma}$ at characteristic points in $\mathbb{CP}^n$. This surface turns out to be the same $\hat{\Sigma}$ and that is how singularities of U_Γ propagate from characteristic points "at infinity" to finite $\mathbb{C}^n$.

Therefore, one may try to show that non-characteristic complex hyperplanes are the only surfaces in $\mathbb{C}^n$ which do not have "characteristic points at ∞" i.e. whose imbeddings in $\mathbb{CP}^n$ still do not have characteristic points.

IV. The function V_Γ defined by (5.5), which we call the <u>modified Schwarz potential</u> (m.s.p.) has been encountered in several places in this paper (cf. (3.1*) and §2 III, IV). It satisfies the Cauchy problem

$$\begin{cases} \Delta V_\Gamma = 1 & \text{near } \Gamma \\ V_\Gamma \equiv 0 & \text{on } \Gamma \end{cases} \qquad (5.6)$$

Although it is apparently just a trivial modification of U_Γ, V_Γ is the more natural form to use in certain applications, especially regarding quadrature domains. It is easy to see that theorem 4.2 can be stated in the following equivalent form:

Theorem 4.2′ *If* Ω *is as in Theorem 4.2, there is a function* V *on* $\mathbb{R}^n$ *satisfying*

$$\Delta V = \chi_\Omega - T \quad \underline{\text{in the distributional sense}} \qquad (5.7)$$

$$V(x) = 0, \; x \in \mathbb{R}^n \setminus \Omega \qquad (5.8)$$

Conversely, if such a V *exists* Ω *is a quadrature domain and (4.2) holds.*

Note that, from (5.7), $V \in C^1(\mathbb{R}^n \setminus \text{supp } T)$ so (5.8) implies that V and grad V tend to zero uniformly as $x \in \Omega$ tends to $\partial\Omega$. It is not hard to show conversely, that *if* V *is any function in* $C^1(\overline{\Omega}) \cap C^2(\Omega \setminus K)$ *(where* $K \subset \Omega$ *is compact) satisfying*

$$\Delta V = 1 \quad \text{on} \quad \Omega \setminus K \qquad (5.9)$$

and

$$V(x) \text{ and grad } V(x) \to 0 \text{ as } x \in \Omega \text{ tends to } \partial\Omega \qquad (5.10)$$

then, when V *is extended to* $\mathbb{R}^n$ *by defining it as* 0 *on*

$\mathbb{R}^n \setminus \Omega$, *the resulting function satisfies (5.7) and (5.8), for a suitable distribution* T *with support in* $K_\epsilon = \{x : \text{dist}(x,K) \leq \epsilon\}$ *(*$\epsilon > 0$ *being arbitrary).*

Note that, in case Γ is a non-singular real-analytic portion of $\partial\Omega$ which is in the closure of $\mathbb{R}^n \setminus \overline{\Omega}$, $V|_\Omega$ extends harmonically across Γ and this extended function equals V_Γ on some neighborhood of Γ. Thus, e.g. if Ω is bounded by a real-analytic nonsingular hypersurface it is a quadrature domain, and near $\partial\Omega$ the function V in Theorem 4.2′ equals $V_{\partial\Omega}$ at points of Ω and 0 at points of $\mathbb{R}^n \setminus \Omega$ (cf. (4.14)).

An important extension of the notion of quadrature domains due to Sakai [Sa 1] and developed further by Gustafsson [Gu 2] is that of q.d. relative to <u>subharmonic</u> test functions, i.e. domains Ω such that in place of (4.2) we have the stronger

$$\int_\Omega u \, dx \geq \langle T, u \rangle \tag{5.11}$$

for every integrable <u>subharmonic</u> function u on Ω that is C^∞ on a neighborhood of supp T (usually in these works T is a positive measure, and the regularity requirement on u can be relaxed). Sakai showed, with suitable regularity hypotheses that, in the setting of Theorem 4.2′, *(5.11) is equivalent to the existence of* V *satisfying (5.7), (5.8)* <u>*and*</u>

$$V(x) \geq 0, \quad \text{all } x \in \mathbb{R}^n \tag{5.12}$$

Moreover, Ω *is then the unique domain satisfying (4.2).* (For an elegant and very detailed treatment of all this see [Gu 2].) Conditions like (5.9), (5.10) arise very naturally when the domain Ω is constructed as the coincidence set in a variational inequality.

In other recent developments, so far unpublished, due to Lavi Karp, H. Shahgholian and one of the present authors (H.S.S.), Theorem 4.2′ has been extended to unbounded domains. This involves essential difficulties because if Ω is unbounded, the existence of a function V with the properties asserted in Theorem 4.2′ does not imply that Ω is a quadrature domain. If however V also satisfies (5.12) then Ω can be shown to be a q.d., even in the strong sense that (5.11) holds.

Karp [Ka 1] has shown that *if* Ω *is an unbounded* q.d., *i.e. (4.2) holds for some* T *with* supp T *compact and* $\subset \Omega$, *then a function* V *exists having the properties asserted in Theorem 4.2′ and*

$$V(x) = O\left(|x|^2 \log |x|\right), \quad x \to \infty \tag{5.13}$$

This improves an earlier result in [Sh 1]. Moreover, if Ω is a "null quadrature domain" (T = 0), (5.13) can be sharpened to

$$V(x) = O\left(|x|^2\right), \ x \to \infty \tag{5.14}$$

Possibly (5.14) holds for all q.d. Karp also showed that if, for an unbounded domain Ω, a function V exists satisfying

(5.9), (5.10) and

$$V(x) = o\ (|x|^3), \quad x \to \infty, \tag{5.15}$$

then Ω is a quadrature domain (and hence V actually satisfies the stronger estimate (5.13)).

The proof of the latter assertion is an easy deduction from Green's formula (cf. §4) plus the following approximation theorem.

<u>Theorem 5.5</u> *(Karp) Let $\Omega \subset \mathbb{R}^n$ be any open set with $\overline{\Omega} \neq \mathbb{R}^n$. Then, the set of functions harmonic in Ω and satisfying*

$$u(x) = O\ (|x|^{-n-1}), \quad x \to \infty \tag{5.16}$$

is dense in $L^1_h\ (\Omega)$.

In general, the harmonic functions that are $O\ (|x|^{-m})$, where $m > n + 1$ do <u>not</u> span $L^1_h\ (\Omega)$, for instance they do not if $\Omega = \{x : |x| > 1\}$. It seems to be an open question to give sufficient (and hopefully necessary, or "nearly so") conditions on Ω that guarantee the density in $L^1_h\ (\Omega)$ of the harmonic functions which are $O(|x|^{-m})$ at infinity, when $m > n + 1$. In [Sh 1] the corresponding problem for analytic functions in $\mathbb{R}^2 = \mathbb{C}$ was solved: *if $\mathbb{R}^2 \setminus \Omega$ contains a continuum going to ∞ then for each m, analytic functions that are $O(|z|^{-m})$ at infinity are dense in $L^1_a(\Omega)$.* Possibly in $\mathbb{R}^n$ an analogous Theorem holds, a relevant sufficient condition being that

$\mathbb{R}^n \setminus \overline{\Omega}$ is suitably "thick" at infinity, in the sense of Essén *et al* [EHLS], Sakai [Sa3], or in other words that infinity be a regular boundary point of Ω for Dirichlet's problem. If this were true, then for such domains the condition

$$V(x) = O(|x|^m) \quad \underline{\text{for some}} \quad m > 0, \quad x \to \infty \tag{5.17}$$

in place of (5.15), would imply that Ω is a quadrature domain (and hence (5.13)).

The above discussion shows the motivation for estimating a function V in a domain Ω which satisfies (5.9) and (5.10) or, in place of (5.9) an equation

$$\Delta V = f \quad \text{on} \quad \Omega \tag{5.18}$$

where $f \in L^\infty(\Omega)$. In case $\Omega = \mathbb{R}^n$, Lavi Karp has shown there is a solution with

$$|V(x)| \leq C_n \; (1 + |x|)^2 \log(2 + |x|) \; \|f\|_\infty \tag{5.19}$$

and this estimate is in general the best possible. The solution satisfying (5.19) is not unique, but any two solutions differ by a quadratic harmonic polynomial. In particular, the choice $f = \chi_\Omega$ allows one to define a reasonable notion of potential of a uniform mass distribution on an open set of $\mathbb{R}^n$ (modulo quadratic harmonic polynomials). All of these potentials have the same singularities, so the "Herglotz problem" (cf. §1) is meaningful.

Cafarelli gave local estimates for functions V satisfying (5.9), (5.10) with minimal regularity assumptions on $\partial\Omega$. These are important in studying the regularity of free boundaries produced by solving variational inequalities, cf. Friedman [Fr, Chapter 2]. For our purposes here, global estimates (when Ω is unbounded) are of greatest interest. One can prove the lower estimate:

Theorem 5.6 *If* V *satisfies (5.9) and (5.10), then*

$$\limsup_{\substack{x\to\infty \\ x\in\Omega}} \frac{|V(x)|}{|x|^2} > 0 \tag{5.20}$$

No upper bound is possible for V under the hypotheses of Theorem 5.2, as simple examples show. If however $V(x) \geq 0$, then one can show (5.14) holds, and even

$$V(x) \leq C\, d(x)^2 \tag{5.21}$$

where $d(x) = \text{dist}\,(x, \partial\Omega)$.

One can also prove (5.21) under other hypotheses. We give here such a proof, since it is very simple.

Theorem 5.7 *Let* $\Omega \subset \mathbb{R}^n$ *be an unbounded convex set, and* $V \in C^1(\mathbb{R}^n)$, $f \in L^\infty(\mathbb{R}^n)$ *satisfy* (5.18) (5.8) *and*

$$V \text{ is a tempered distribution on } \mathbb{R}^n \tag{5.22}$$

Then,

$$V(x) \leq C\, \|f\|_\infty^2 \cdot d(x)^2 \tag{5.23}$$

where C *is an a constant depending on* V *but not on* x.

The proof requires

<u>Lemma 5.8</u> *For sufficiently large* $N = N(n)$ *the function*

$$K(\xi) = \frac{(e^{i\xi_1} - 1)^N}{|\xi|^2} \tag{5.24}$$

is in FL^1, *i.e. is the Fourier transform of some function* $k \in L^1 (\mathbb{R}^n)$.

We sketch the proof: it is enough to check that $K|_B$ is in $FL^1|_B$ and $K|_{B'}$ is in $FL^1|_{B'}$ where $B = \{|x| < 1\}$, $B' = \{|x| > \frac{1}{2}\}$. For the former assertion it suffices to check that $K|_B \in C^r (B)$ where r can be made as large as desired (and hence $> n/2$) which by Bernstein's theorem implies $K|_B$ is in $FL^1|_B$) by choosing N large. On B', $|\xi|^{-2}$ coincides with an element of FL^1 (this is well known) and $(e^{i\xi_1} - 1)^N$ is the Fourier transform of a bounded measure so the product of these is in $FL^1|_{B'}$. This proves the lemma.

<u>Proof of Theorem</u>. From (5.18), taking Fourier transforms we have $|\xi|^2 \hat{V} (\xi) = \hat{f} (\xi)$ (We emphasize that throughout this paper all equations and Fourier transforms are in the sense of distribution theory). Choose N so large that the conclusion of the lemma holds, and also so that K has sufficient regularity that

$\hat{V}(\xi)K(a\xi)$ $(a > 0)$ is a well-defined distribution on a neighborhood of 0.

Hence, with $K = \hat{k}$ given by (5.24) and $a > 0$

$$\hat{V}(\xi)\,\hat{k}(a\xi)\,|\xi|^2 = \hat{f}(\xi)\,\hat{k}(a\xi)$$

or,

$$\hat{V}(\xi)\,(e^{ia\xi_1} - 1)^N = \hat{f}(\xi)\,\hat{k}(a\xi)\,a^2.$$

Now, take the norms of both sides in the space FL^{∞}, where $\|\hat{g}\|_{FL^p}$ designates $\|g\|_{L^p}$ for any $g \in L^p = L^p(\mathbb{R}^n)$. We get

$$\|\hat{V}(\xi)(e^{ia\xi_1} - 1)^N\|_{FL^{\infty}} = a^2\|\hat{f}(\xi)\hat{k}(a\xi)\|_{FL^{\infty}}$$

$$\leq a^2\,\|\hat{f}\|_{FL^{\infty}}\,\|\hat{k}(a\xi)\|_{FL^1} = a^2\,\|f\|_{\infty} \bullet \|k\|_1.$$

The term on the left side equals

$$\sup_{x\in\mathbb{R}^n}\Big|\sum_{j=0}^{n}(-1)^j\binom{n}{j}\,V\,(x + ja\xi)\Big|,$$

where $\xi = (1,0,\ldots 0)$. It is clear that we could have taken in place of this vector ξ any unit vector (with slight complication of notations) so we conclude that there is an absolute constant C such that

$$\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j} V(x + ja\xi)\right| \leq C\ \|f\|_\infty\ a^2$$

holds for $x \in \mathbb{R}^n$ and any $a > 0$ and unit vector ξ. Choosing $x \in \Omega$ and a, ξ so that $x + a\xi$ is the point of $\partial\Omega$ nearest to x we immediately get (5.23) (since all $V(x + ja\xi)$ with $j \geq 1$ are 0 in view of the convexity of Ω), and this finishes the proof.

Remarks. By a similar, but more delicate use of this technique one can prove the result of Karp embodied in (5.19) see [Sh 3].

The above technique also serves to establish (5.14) (which of course is weaker than (5.23)) under much weaker conditions than convexity, it suffices e.g. if Ω is contained in a half-space.

Combining Theorems 5.6 and 5.7 gives

Corollary 5.9 *If* V *satisfies the hypotheses of Theorem 5.7,* Ω *must be "very large" in the sense: there is a number* $\rho > 0$ *and a sequence* $\{x^j\} \subset \Omega$ *with* $x^j \to \infty$ *such that all the balls*

$$B_j := \{x : |x - x^j| < \rho\ |x^j|\}$$

are contained in Ω.

In particular, every unbounded, convex quadrature domain has this property. Thus, e.g. the domain bounded by a cylinder or paraboloid cannot be a q.d. (We emphasize that we mean this here in the sense that T in (4.2) has compact support; Gustafsson and Sakai allow T to have support extending to $\partial\Omega$

(or, to ∞ if Ω is unbounded)). Another type of Theorem asserting that unbounded quadrature domains are "very large", with no convexity assumption, was given by Sakai for $n = 2$ [Sa 1, p. 90] and extended to $\mathbb{R}^n$ by Shahgholian [Sha 3].

In the case of "null quadrature domains", i.e. $\int_\Omega u = 0$ for all $u \in L^1_h(\Omega)$ (such Ω are necessarily unbounded) more precise statements can be made. Sakai [Sa 2] has described all such domains for $n = 2$. Shahgholian has observed that if $\{E_j\}$ is a sequence of domains, each of which is the interior of an ellipse, then $\mathbb{R}^2 \setminus \bigcup_{j=1}^{\infty} E_j$ is a null q.d., and that all of Sakai's domains can be attained in this way. In $\mathbb{R}^n$ the analogous construction (where the E_j are ellipsoids) still yields null q.d. and Shahgholian raises the question whether all null q.d. can be constructed in this way, a tantalizing problem. If the answer were "yes", a (weak) corollary would be: <u>the complement of a null q.d. in</u> $\mathbb{R}^n$ <u>is convex</u>. One of the present authors (H.S.S., unpublished) has shown, subject to some regularity hypotheses that this is true and moreover the associated function V is nonnegative on Ω (so that $\int_\Omega u \, dx \geq 0$ holds for every subharmonic integrable u).

V. Modified Schwarz potentials lead to an interesting relation between the Cauchy problem for the Laplace operator and integral geometry (this relation exists for all linear partial differential operators but we restrict our discussion to the Laplace operator, where it is particularly simple.) To illustrate it, consider first the Cauchy problem (in $\mathbb{R}^2$)

$$\begin{cases} \Delta u = f & \text{near } \Gamma \\ u \equiv 0 & \text{on } \Gamma \end{cases} \tag{5.25}$$

where Γ is a non-singular analytic arc, f is real-analytic on a neighborhood of Γ, and the figure D bounded by Γ and the chord joining its endpoints is convex. We shall assume that f and the solution u of (5.25) are analytic in D, which may always be achieved by restricting to a sufficiently small portion of Γ. Consider now a chord joining two points of Γ; together with Γ it bounds a domain $D_o \subset D$. Let v denote the modified Schwarz potential of the chord; if the equation of the latter is

$$a_1 x_1 + a_2 x_2 - t = 0$$

with $a_1^2 + a_2^2 = 1$, we have $v(x) = \frac{1}{2}(a_1 x_1 + a_2 x_2 - t)^2$ and by Green's formula (note that the boundary terms vanish)

$$0 = \int_{D_o} u \, \Delta v \, dx - \int_{D_o} v \, \Delta u \, dx$$

$$= \int_{D_o} u \, dx - \int_{D_o} f \, v \, dx, \text{ so}$$

$$\int_{D_o} u \, dx = \frac{1}{2} \int_{D_o} (a_1 x_1 + a_2 x_2 - t)^2 f(x) \, dx \qquad (5.26)$$

Thus, $\int_{D_o} u \, dx$ is known from the data of the problem. By varying the portion of the chord bounding D_o we can compute a whole family of such integrals, and the "integral geometry" version of (5.25) is to find u from these integrals. This seems to be a very interesting problem in its own right. Thus far we have not studied methods for solving it numerically, but wish to make a few observations.

First of all, it is easy to show that there is at most one function u real-analytic on a neighborhood of Γ satisfying all the relations (5.26). (Concerning uniqueness with weaker regularity hypotheses, we don't have the answer.)

Secondly, by varying t in (5.26) and differentiating, it is easy to deduce the value of the integral of u with respect to arc length along every chord with endpoints on Γ. Thus, this information can be taken as the "input" to the integral geometry problem, which now has formal similarity with the basic problem of mathematical tomography, Radon transforms etc.

Thirdly, the convexity assumption about Γ can be dispensed with if, in place of domains like D_o we use domains bounded by Γ and a circle with center lying off Γ. The role of v is now played by the m.s.p. of this circle (moreover, the center can be taken on Γ, as a simple limiting procedure shows).

The extension to higher dimensions (using hyperplanes instead of lines, etc.) of the above formal considerations is straightforward, but here, as with other aspects of our subject, essential difficulties remain.

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